

Unique Covering problems with Geometric Sets

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Set Cover

For a set system $(U, \mathcal{F})$ where U is a finite universe of n elements and $\mathcal{F}$ is a family of subsets of U , is there a sub family of at most k sets in $\mathcal{F}$ whose union is U .

We say that an element x in U is *covered* by a set S from $\mathcal{F}$ if the set S contains the element x .

Covering Uniquely

All or some of the elements are covered **uniquely**.

- ▶ All elements must be covered uniquely by at most k sets. (EXACT COVER)
- ▶ All elements must be covered, *and* at least k elements must be covered uniquely. (UNIQUE SET COVER)
- ▶ At least k elements are covered uniquely. (UNIQUE COVER)

We study the parameterized complexity of these problems in a geometric setting.

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- ▶ A **kernelization** algorithm for a parameterized problem Π is a polynomial time procedure which takes as input an instance (x, k) , where k is the parameter, and returns an instance (x', k') such that $(x, k) \in \Pi$ if and only if $(x', k') \in \Pi$ and $|x'| \leq g(k)$ and $k' \leq h(k)$, for some computable functions g, h . The returned instance (x', k') is said to be the **kernel** for the instance (x, k) of Π .

Results

	Exact Cover		Unique Cover	Unique Set Cover
Parameter	Size of solution		Number of elements uniquely covered	
Abstract Sets	W[1]-hard		FPT	W[1]-hard
			Quadratic Element Kernel	
Lines	FPT	VC-Dimension	FPT	FPT
	Quadratic Kernel		Quadratic Kernel	Poly Kernel
Hyperplanes $\mathbb{R}^d$	$k^{O(d^2)}$ kernel		$k^{O(d)}$ Instance Kernel (Quadratic Element Kernel)	$k^{O(d^2)}$ kernel
Unit Squares	W[1]-hard		Poly Kernel	Open

Exact Cover

EXACT COVER

Parameter: k

Input: A set system $(U, \mathcal{F})$ of n elements and m sets, and a positive integer k .

Question: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k that covers every element of U such that each element is contained in exactly one set in $\mathcal{F}'$?

some literature.....

Exact Cover: Our Results

Lemma (★)

EXACT COVER *on set systems of unit squares* is $W[1]$ -hard.

Lemma

EXACT COVER *on set systems of lines* is FPT, with a polynomial kernel.

Lemma

EXACT COVER *on set systems of hyperplanes* is FPT, with a polynomial kernel.

Unique Cover

UNIQUE COVER

Parameter: k

Input: A set system $(U, \mathcal{F})$ of n elements and m sets, and a positive integer k .

Question: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that, among the set of elements covered by $\mathcal{F}'$, there is a subset $S \subset U$, $|S| \geq k$ with each element of S being contained in exactly one set in $\mathcal{F}'$?

Unique Cover

Lemma (★)

UNIQUE COVER *admits a polynomial kernel for set systems of lines, hyperplanes, axis-parallel rectangles and disks.*

Unique Set Cover

UNIQUE SET COVER

Parameter: k

Input: A set system $(U, \mathcal{F})$ of n elements and m sets, and a positive integer k .

Question: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ that covers every element of U such that there is a subset $S \subseteq U$, $|S| \geq k$ where each element of S is contained in exactly one set in $\mathcal{F}'$?

Unique Set Cover

Lemma (★)

UNIQUE SET COVER *is* $W[1]$ -hard.

Reduction from MULTICOLORED CLIQUE problem.

Unique Set Cover

Lemma (★)

UNIQUE SET COVER *on set systems of lines* is NP-complete.

Reduction from 1-IN-3-SAT

Unique Set Cover

Lemma

UNIQUE SET COVER *for sets of bounded intersection c is FPT when parameterized by $c + k$.*

Unique Set Cover I

- ▶ Construct a minimal set cover $\mathcal{S}$ for the instance.
- ▶ If the size of $\mathcal{S}$ is at least k , then it is a solution for UNIQUE SET COVER
- ▶ Similarly, if there is a set $S \in \mathcal{S}$ that contains at least k private elements, then too $\mathcal{S}$ is a solution for UNIQUE SET COVER

Unique Set Cover II

- ▶ Suppose there are at most $k - 1$ sets in $\mathcal{S}$ and each set has at most $k - 1$ private elements
- ▶ By pigeonhole principle, there must be a set $S \in \mathcal{S}$ which contains at least $n/(k - 1)$ elements of U
- ▶ The number of elements in S that can belong to other sets of $\mathcal{S}$ is at most $c(k - 2)$
- ▶ If $\frac{n}{k-1} - c(k - 2) \geq k$ then the given instance is a YES instance.

Unique Set Cover III

- ▶ Otherwise, $\frac{n}{k-1} - c(k-2) \leq k-1$, which implies that $n \leq (1+c)(k-1)^2$.
- ▶ the number of sets are bounded by $n^{c+1} \leq ((1+c)(k-1)^2)^{c+1}$
- ▶ guess the uniquely covered elements, and the distribution of the uniquely covered elements in a UNIQUE SET COVER solution.
Finally, we can check whether there are sets in $\mathcal{F}$ to validate the guess in polynomial time. As the number of guesses is an FPT function, the running time of this algorithm is FPT

Unique Set Cover

Lemma

UNIQUE SET COVER *on set systems of lines* is FPT.

Lines are set systems with bounded intersection 1

Higher dimensions?

Unique Set Cover

Lemma

UNIQUE SET COVER *on set systems of hyperplanes in $\mathbb{R}^d$* is FPT.

Unique Set Cover

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Correctness of the reduction rule can be shown by induction on i .

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- ▶ At the end, any hyperplane contains at most k^{d-1} points
- ▶ If there are at least $k + 1$ hyperplanes in $\mathcal{G}$, then we correctly say YES

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KERNEL!!