

## *Feedback Vertex Set*

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The immediate simplification is to remove isolated (degree 0) and pendant (degree 1) vertices.

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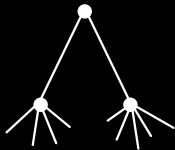
What about vertices of degree two?

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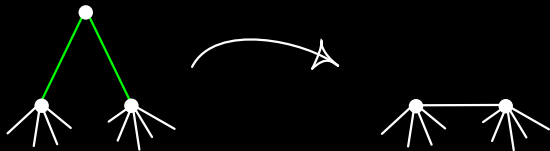
What about vertices of degree two?

These are also simple to handle...

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What about vertices of high degree?

## *Feedback Vertex Set*

Recall that we were lucky with high degree vertices for vertex cover.

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Let's be optimistic and *assume* that we will get lucky again...

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What can we say about a graph that is simplified to the extent of being degree at least three, and not having vertices of degree more than  $s$ ?

# Feedback Vertex Set

$$1.5n \leq \# \text{of edges}$$

*Minimum degree three...*

*FVS*

*forest*

# Feedback Vertex Set

$$1.5n \leq \# \text{of edges} \leq sk + k^2 + (n - k - 1)$$

*FVS*

*forest*

# Feedback Vertex Set

$$1.5n \leq \# \text{of edges} \leq sk + k^2 + (n - k - 1)$$

$$n \leq 2(sk + (k - 1) + k^2)$$

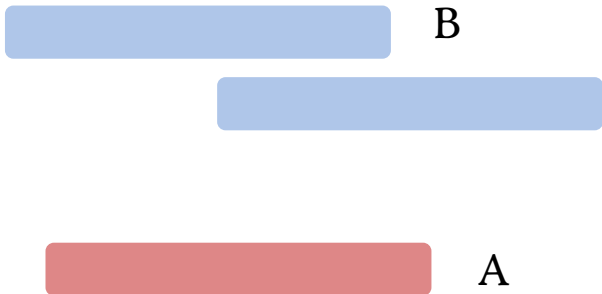
*FVS*

*forest*

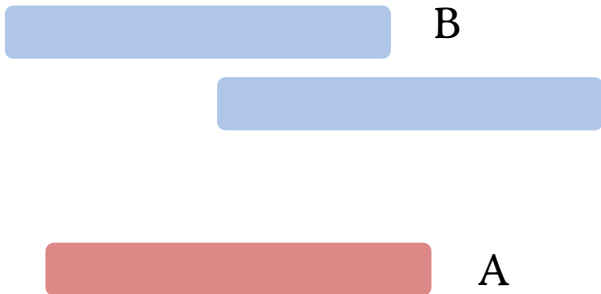
<sup>2)</sup>

## *Feedback Vertex Set*

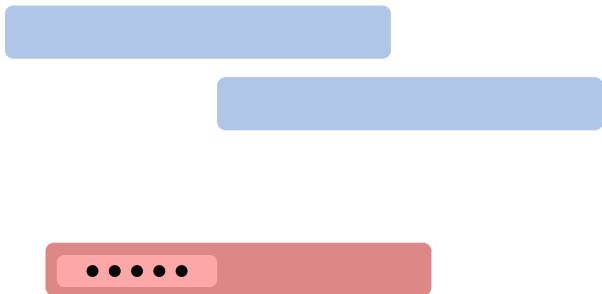
Now we simply have to target high-degree vertices, and try to discover the procedure  $X$  that we assumed before.



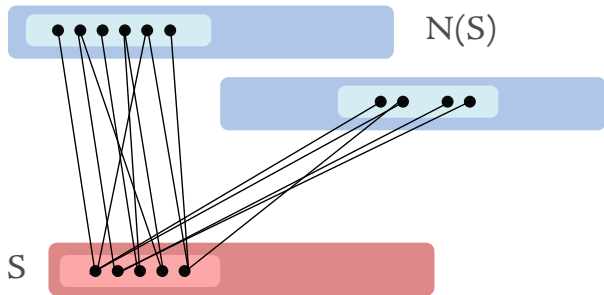
Consider a bipartite graph one of whose parts (say B) is at least twice as big as the other (call this A).



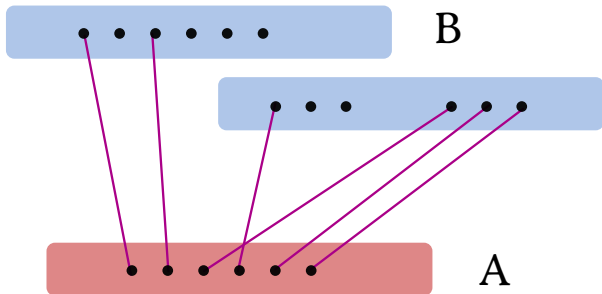
Assume that there are no isolated vertices in B.



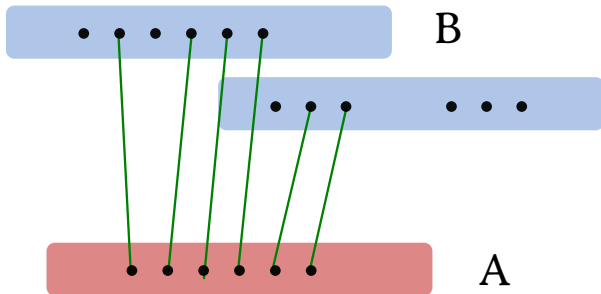
Suppose, further, that for every subset  $S$  in  $A$ ,



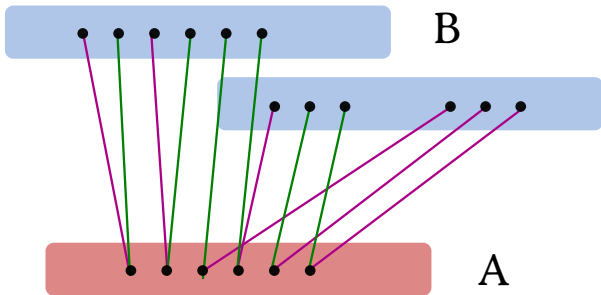
Suppose, further, that for every subset  $S$  in  $A$ ,  $N(S)$  is at least twice as large as  $|S|$ .



Then there exist two matchings saturating  $A$ ,



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Then there exist two matchings saturating A,  
and disjoint in B.

Claim:

If  $N(A) \geq 2|A|$ , then there exists a subset  $S$  of  $A$  such that:

there 2 matchings saturating the subset  $S$   
that are vertex-disjoint in  $B$ .

Claim:

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that are vertex-disjoint in  $B$ .

Claim:

If  $|B| \geq 2|A|$ , then there exists a subset  $S$  of  $A$  such that:

there 2 matchings saturating the subset  $S$   
that are vertex-disjoint in  $B$ ,

provided  $B$  does not have any isolated vertices.

If  $N(S) \geq 2|S|$  for all subsets  $S$ , then  
there are 2 matchings,  
vertex-disjoint in  $B$ , saturating  $A$ .

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In this case,  $S = A$  and we are done.

If  $N(S) \geq 2|S|$  for all subsets  $S$ , then  
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Else: there exists a subset  $S$  for  
which  $N(S)$  is smaller than  $2|S|$ .

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Throw it away.

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Throw it away.

Delete the vertices of  $S$  and the edges incident on it,

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there are 2 matchings,  
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Else: there exists a subset  $S$  for  
which  $N(S)$  is smaller than  $2|S|$ .

Throw it away.

and remove any isolated vertices created in  $B$ .

If  $N(S) \geq 2|S|$  for all subsets  $S$ , then  
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In the graph that is left,  $|B|$  is still  
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$$N(S) \leq 2|S|$$

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$$|B| - N(S) \geq 2|A| - 2|S|$$

If  $N(S) \geq 2|S|$  for all subsets  $S$ , then  
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In the graph that is left,  $|B|$  is still  
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$$|B| - N(S) \geq 2(|A| - |S|)$$

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Throw it away.

**Done: by repetition.**

$$|B| - N(S) \geq 2(|A| - |S|)$$

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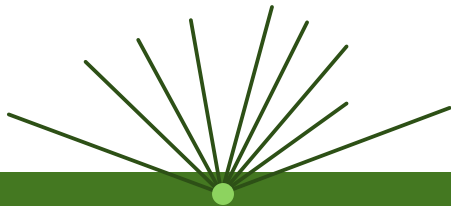
Throw it away.

Done: by repetition.

**(And you'll never clear everything  
out, because  $|B|$  is consistently  
more than  $2|A|$ .)**

$$|B| - N(S) \geq 2(|A| - |S|)$$

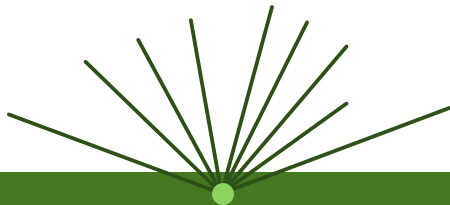
Ingredients



a high-degree vertex,  $v$

# Ingredients

a small hitting set,  
sans  $v$



a high-degree vertex,  $v$

# Ingredients

Given a high-degree vertex  $v$ , finding a small hitting set that does not contain  $v$ .

Given a high-degree vertex  $v$ , finding a small **hitting set** that does not contain  $v$ .

A subset whose removal makes the graph acyclic.

Given a high-degree vertex  $v$ , finding a **small** hitting set that does not contain  $v$ .

A polynomial function of  $k$ .

Given a **high-degree** vertex  $v$ , finding a small hitting set that does not contain  $v$ .

At least twice the size of the hitting set.

Given a high-degree vertex  $v$ , **finding** a small hitting set that does not contain  $v$ .

Find an **approximate** hitting set  $S$ .

Given a high-degree vertex  $v$ , **finding** a small hitting set that does not contain  $v$ .

If  $S$  does not contain  $v$ , we are done.

Given a high-degree vertex  $v$ , **finding** a small hitting set that does not contain  $v$ .

Else:  $v \in S$ . Delete  $S \setminus v$  from  $G$ .

Given a high-degree vertex  $v$ , **finding** a small hitting set that does not contain  $v$ .

The only remaining cycles pass through  $v$ .

Given a high-degree vertex  $v$ , **finding** a small hitting set that does not contain  $v$ .

Find an optimal cut set for paths from  $N(v)$  to  $N(v)$ .

Given a high-degree vertex  $v$ , **finding** a small hitting set that does not contain  $v$ .

Why is this cut set small enough?

Given a high-degree vertex  $v$ , finding a small hitting set that does not contain  $v$ .

When is this cut set small enough?

Given a high-degree vertex  $v$ , finding a small hitting set that does not contain  $v$ .

When is this cut set small enough?

When the largest collection of vertex disjoint paths from  $N(v)$  to  $N(v)$  is small.

Given a high-degree vertex  $v$ , **finding** a small hitting set that does not contain  $v$ .

When is this cut set small enough?

When the largest collection of vertex disjoint paths from  $N(v)$  to  $N(v)$  is *not* small...

Given a high-degree vertex  $v$ , **finding** a small hitting set that does not contain  $v$ .

When is this cut set small enough?

When the largest collection of vertex disjoint paths from  $N(v)$  to  $N(v)$  is *not* small... we get a reduction rule.

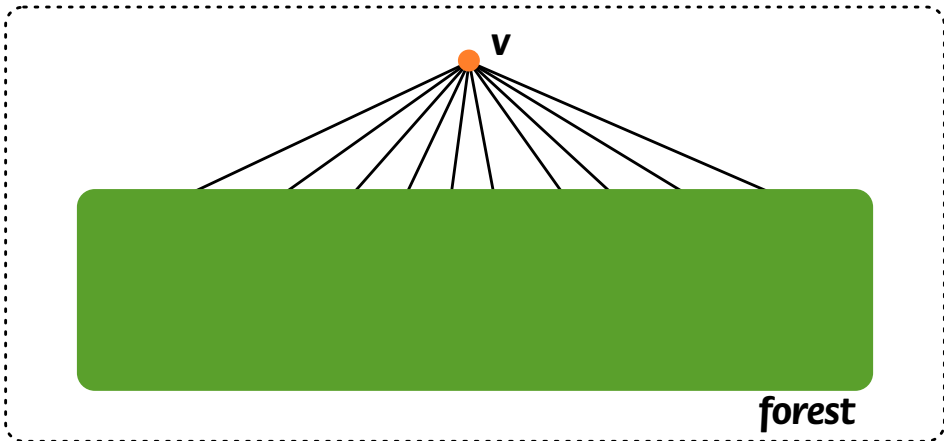
Given a high-degree vertex  $v$ , **finding** a small hitting set that does not contain  $v$ .

When is this cut set small enough?

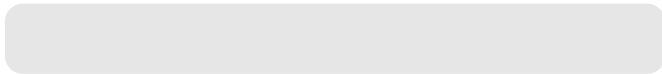
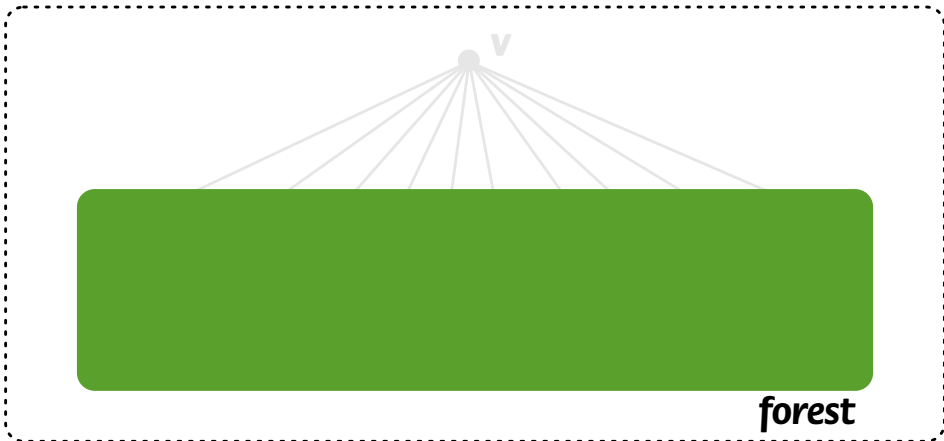
More than  $k$  vertex-disjoint paths from  $N(v)$  to  $N(v)$   
→  $v$  belongs to *any* hitting set of size at most  $k$ .

Given a high-degree vertex  $v$ , finding a small hitting set that does not contain  $v$ .

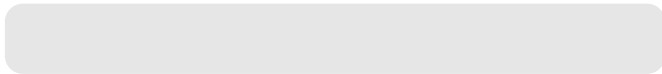
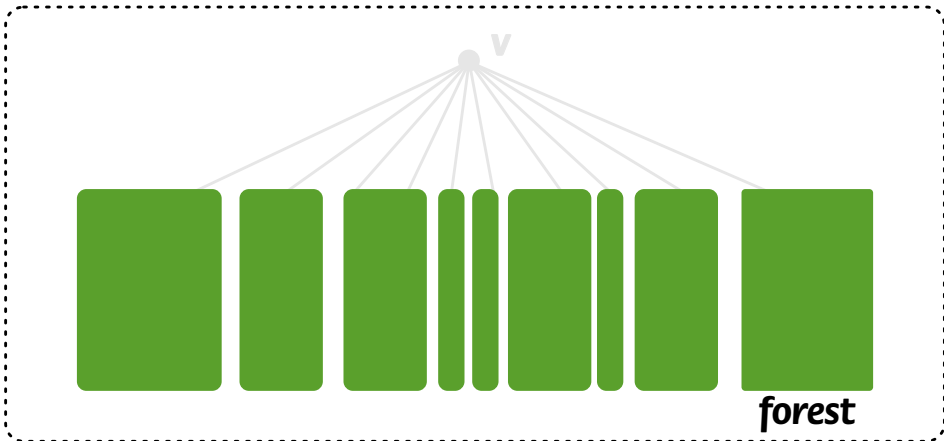
So either  $v$  “forced”, or we have hitting set of suitable size.  
Notice that we arrive at either situation in polynomial time.



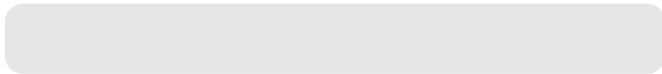
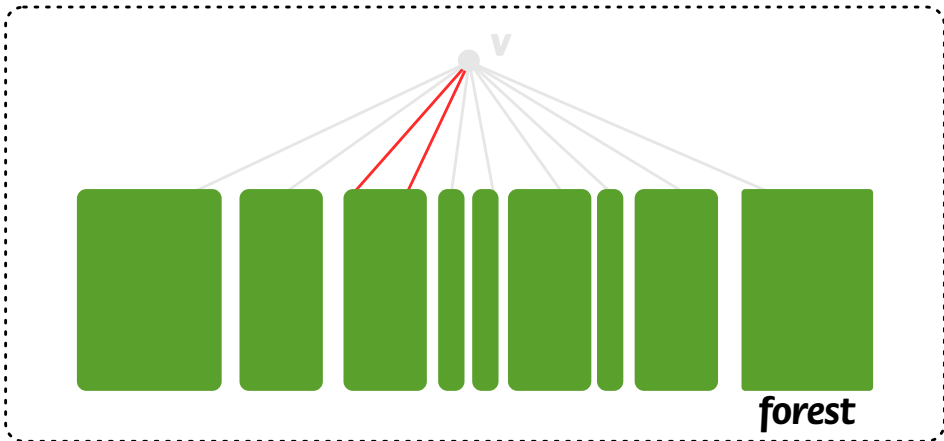
***hitting set that excludes  $v$***



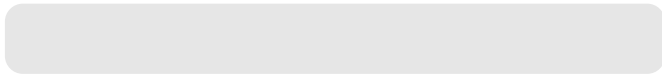
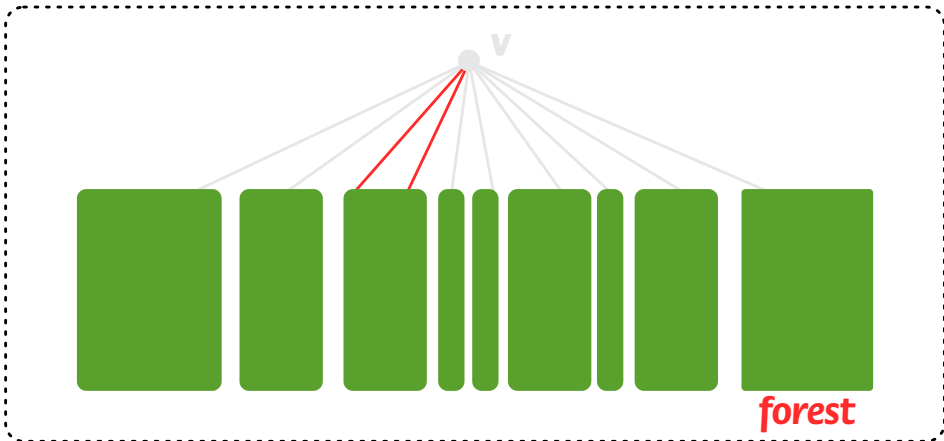
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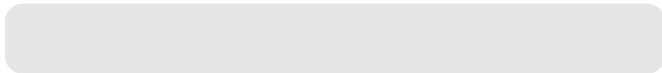
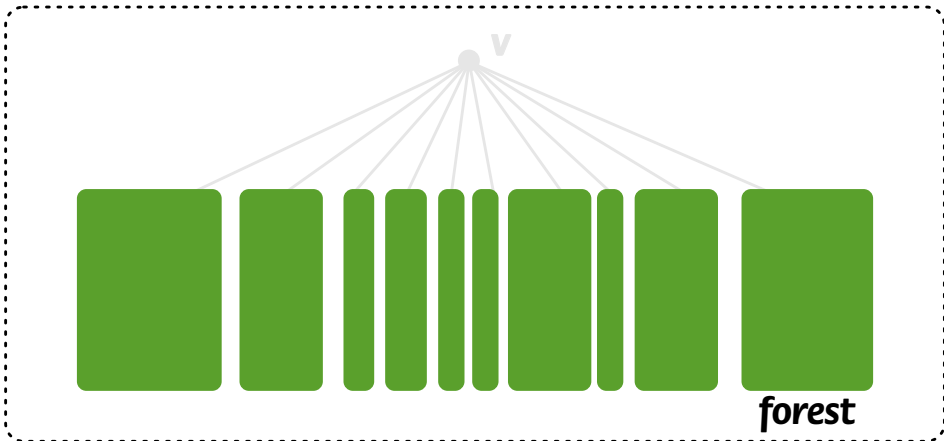
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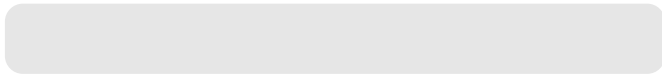
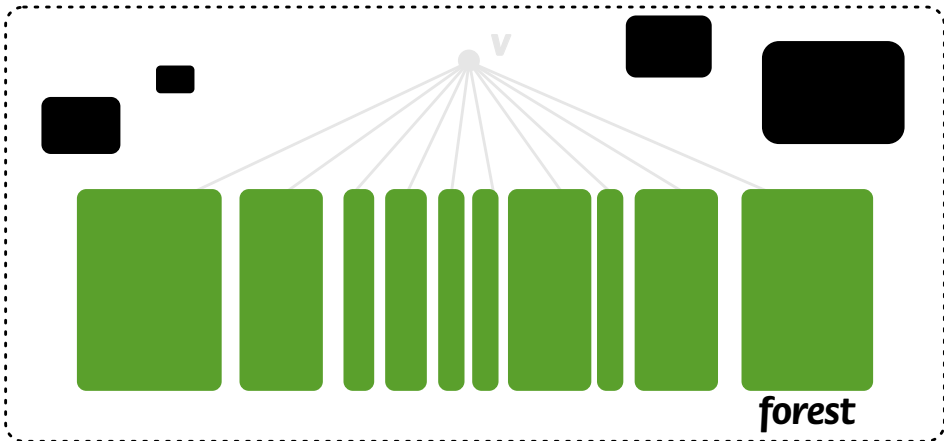
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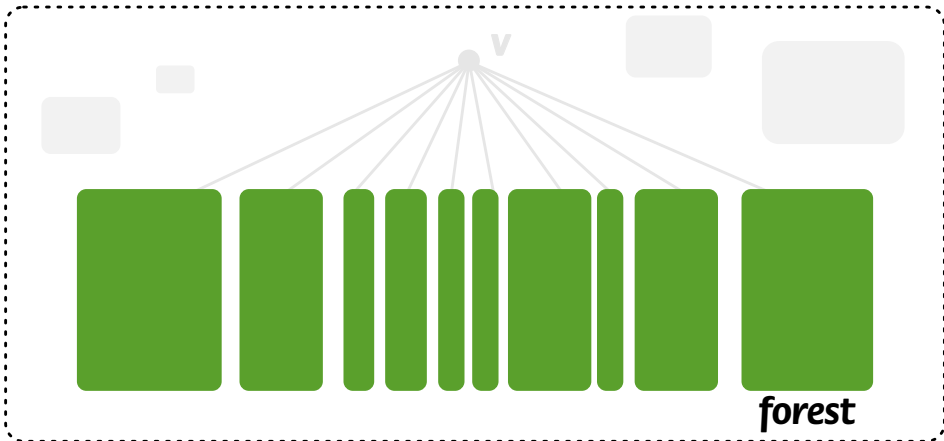
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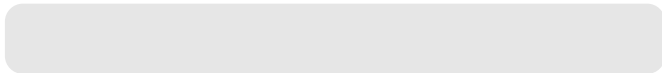
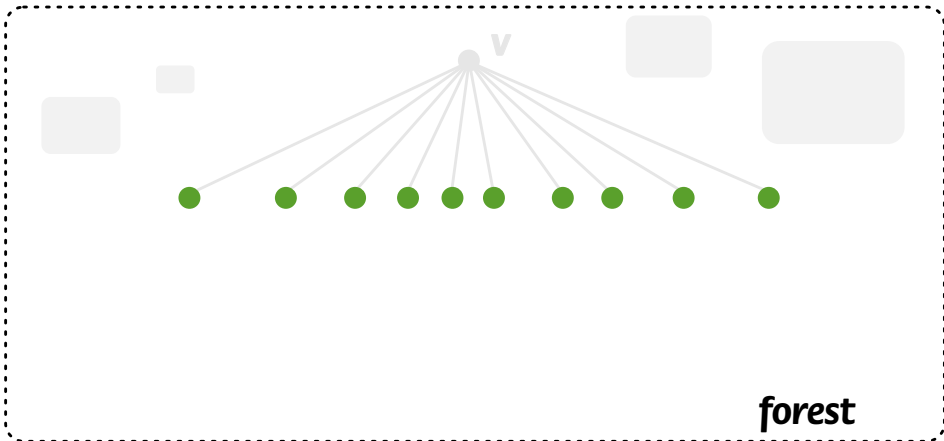
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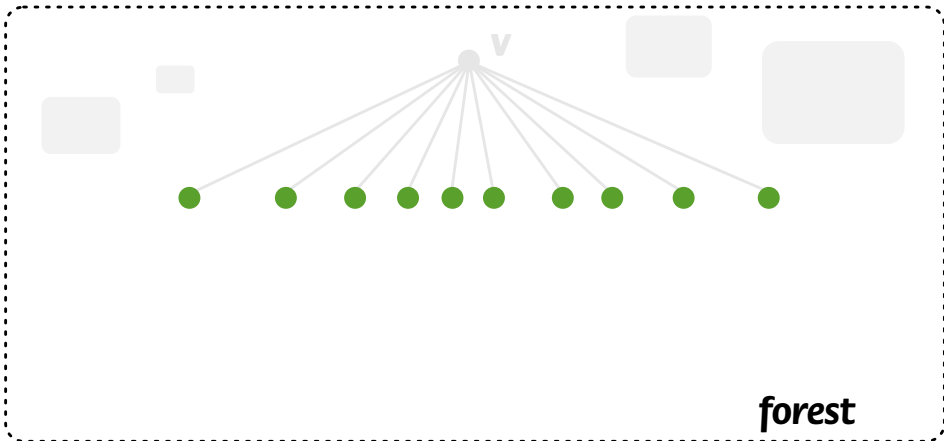
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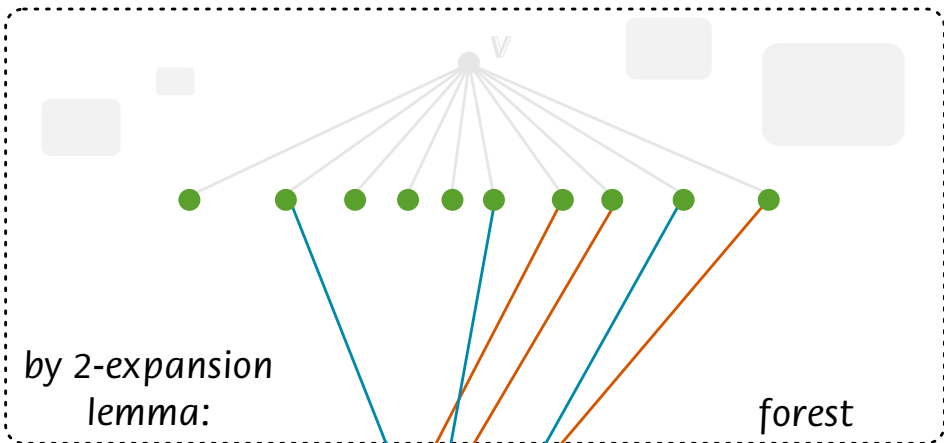
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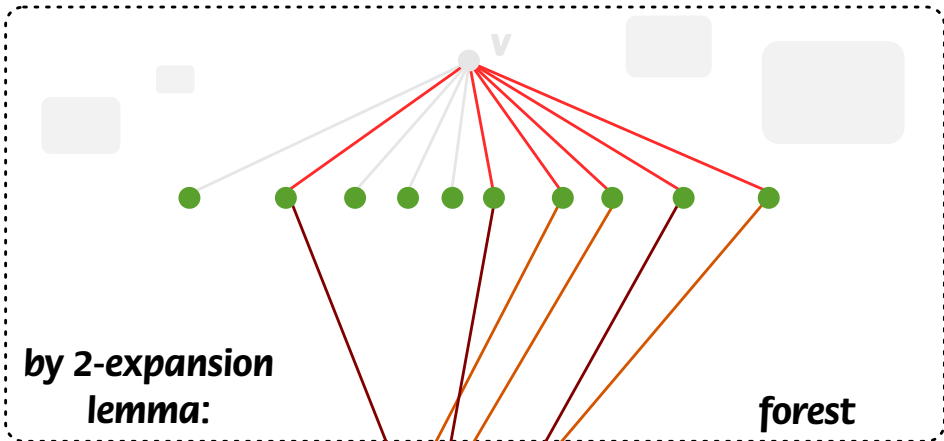
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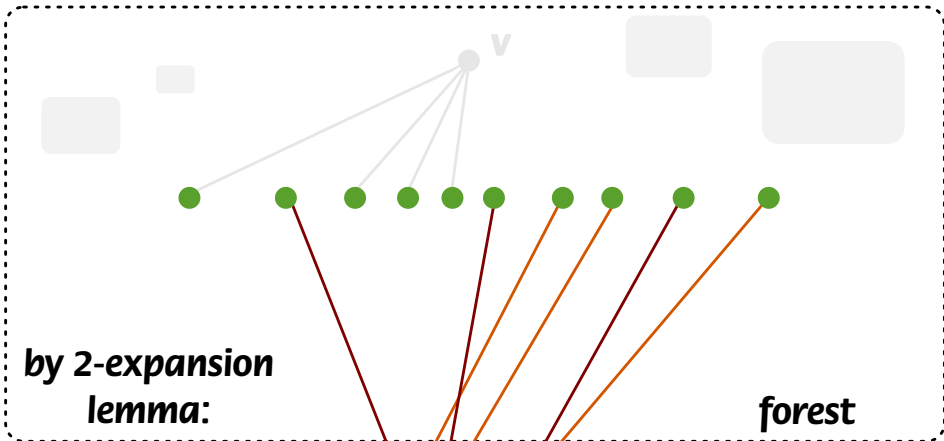
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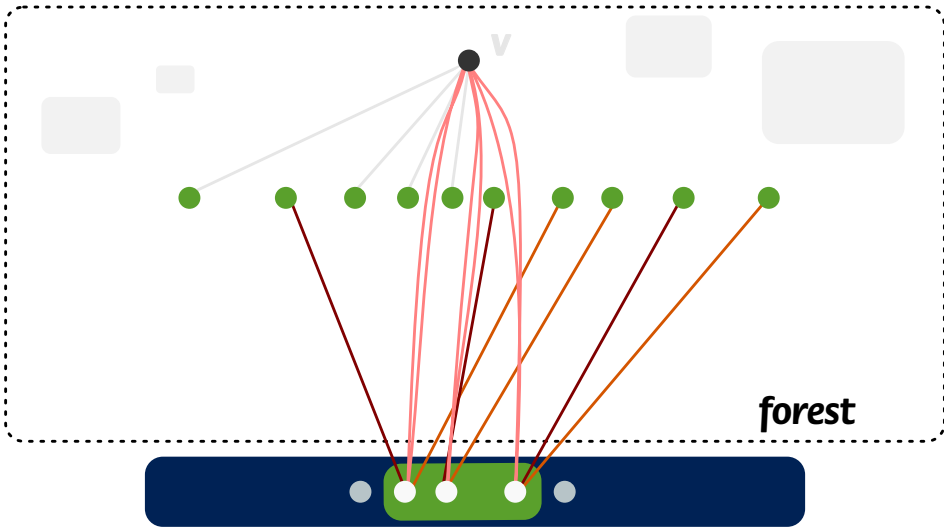
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**hitting set that excludes  $v$**



hitting set that excludes  $v$



***hitting set that excludes  $v$***

## The Forward Direction

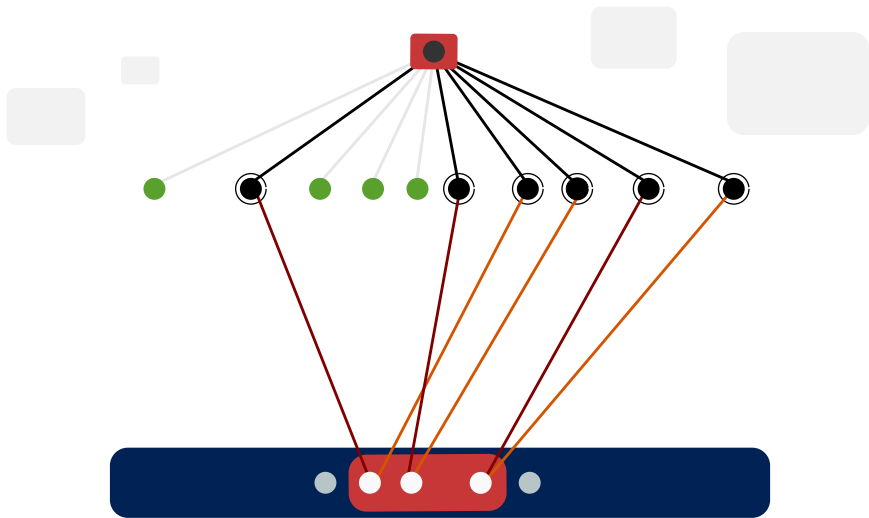
The Forward Direction

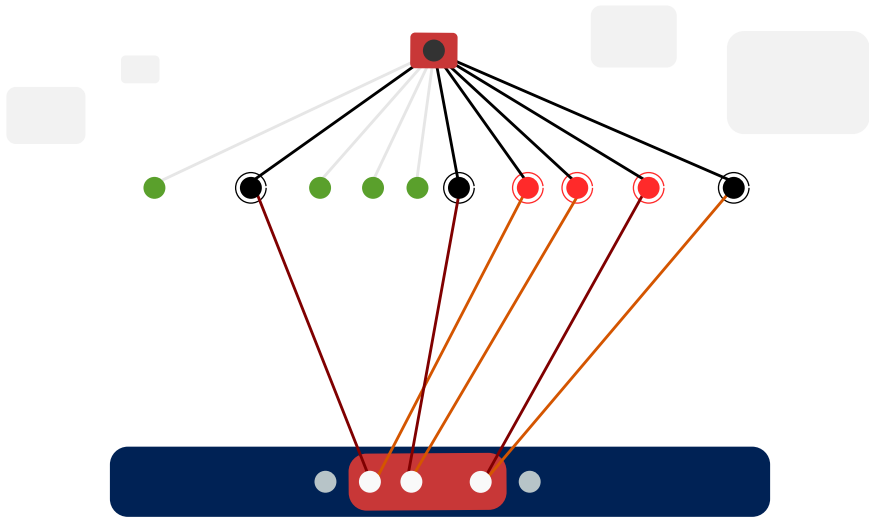
$$\text{FVS} \leq k \text{ in } G \Rightarrow \text{FVS} \leq k \text{ in } H$$

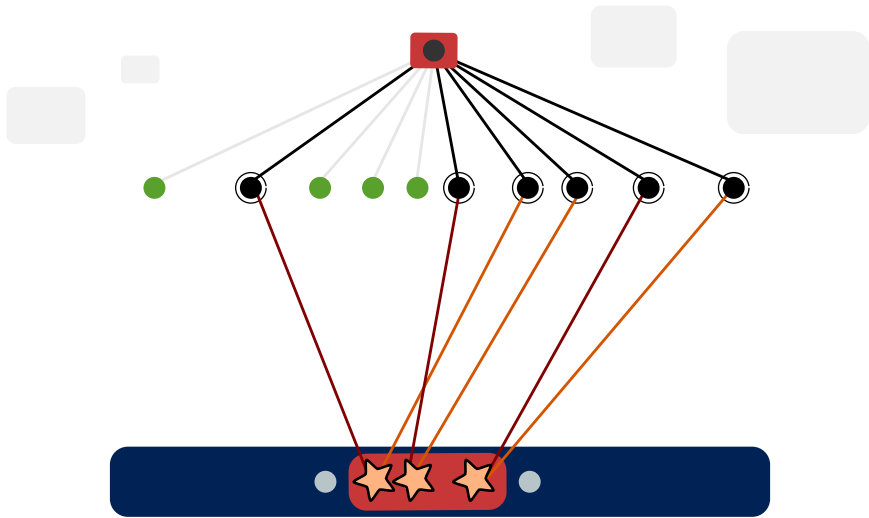
If  $G$  has a FVS that either contains  $v$  or all of  $S$ , we are in good shape.

Consider now a FVS that:

- Does not contain  $v$ ,
- and omits at least one vertex of  $S$ .







Notice that this does not lead to a larger FVS:

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For every vertex  $v$  in  $S$  that a FVS of  $G$  leaves out,

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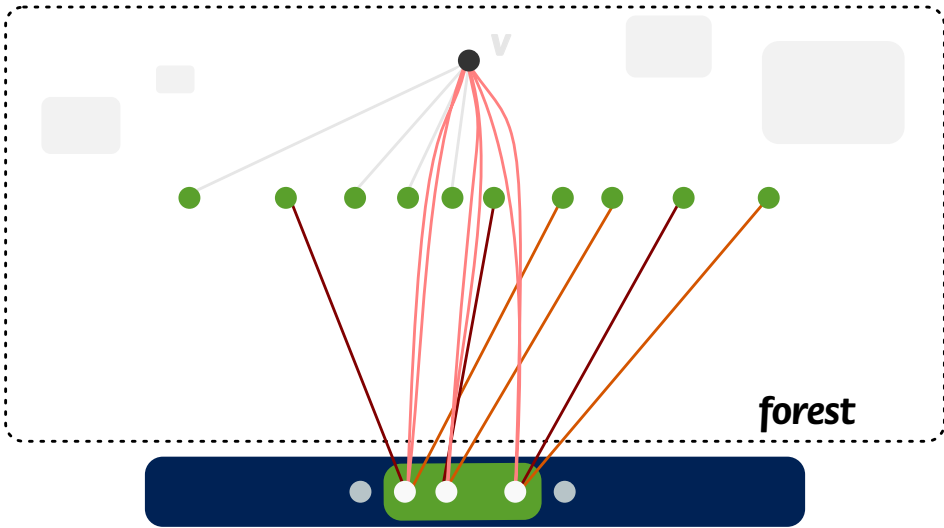
For every vertex  $v$  in  $S$  that a FVS of  $G$  leaves out,

it must pick a vertex  $u$  that kills no more than all of  $S$ .

## The Reverse Direction

The Reverse Direction

$$\text{FVS} \leq k \text{ in } G \Leftarrow \text{FVS} \leq k \text{ in } H$$



***hitting set that excludes  $v$***

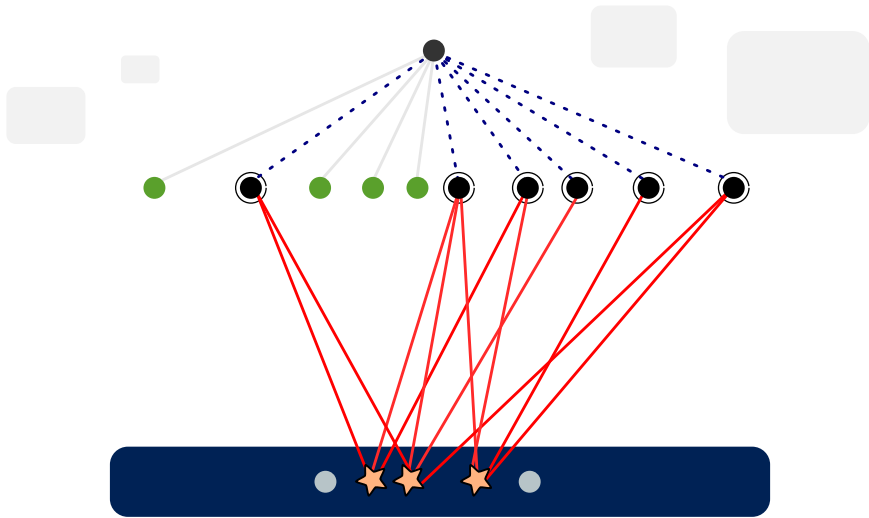
## The Only Danger

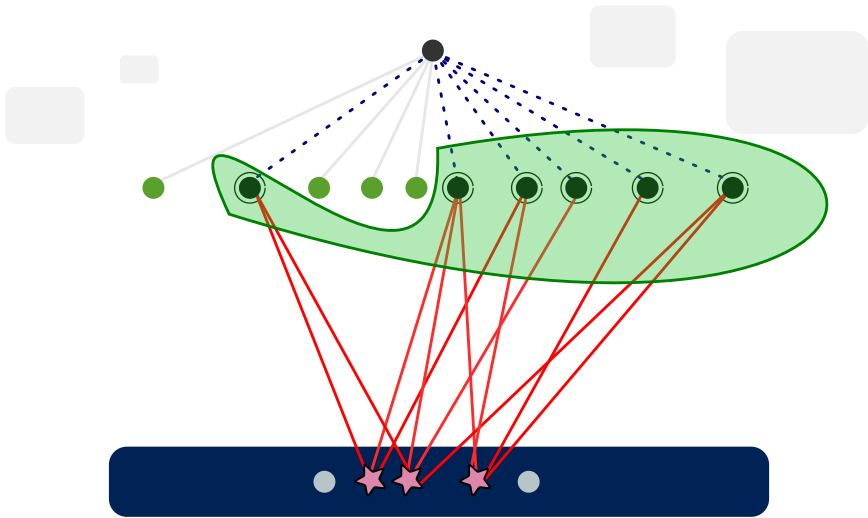
Cycles that:

- pass through  $v$ ,
- non-neighbors of  $v$  in  $H^1$
- and do not pass through  $S$ .

---

<sup>1</sup>neighbors in  $G$ , however





## *Feedback Vertex Set*

Having worked out the details, it turns out that vertices whose degree is more than  $8k$  can be handled, so this leads to an  $O(k^2)$  kernel.