

Expansions for Reductions

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Joint work with
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Talk Outline

A condition clearly necessary for the existence of a matching of A is that every subset of A has enough neighbours in B , i.e. that

$$|N(S)| \geq |S| \quad \text{for all } S \subseteq A.$$

The following *marriage theorem* says that this obvious necessary condition is in fact sufficient:

Theorem 2.1.2. (Hall 1935)

G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.

We give three proofs, ranging from the natural and pedestrian to the slick and elegant. The theorem can also be derived easily from König's theorem (Exercise 4).

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The following *marriage theorem* says that this obvious necessary condition is in fact sufficient:

36

2. Matching, Covering and Packing

Theorem 2.1.2. (Hall 1935) $|N(S)| \geq q|S|$
 G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.

q matchings, mutually disjoint in B

We give three proofs, ranging from the natural and pedestrian to the slick and elegant. The theorem can also be derived easily from König's theorem (Exercise 4).

Talk Outline

q -Expansion Lemma

$$|N(S)| \geq q|S|$$

G contains ~~a matching~~ of A if and only if ~~$|N(S)| \geq |S|$~~ for all $S \subseteq A$.

q matchings, mutually disjoint in B

Talk Outline

q -Expansion Lemma

with applications to Vertex Cover,
Feedback Vertex set,
and beyond.

Part 1

A Tale of Two Matchings



Consider a bipartite graph one of whose parts (say B) is at least twice as big as the other (call this A).

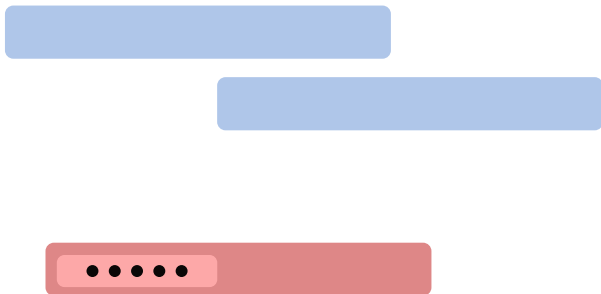


B

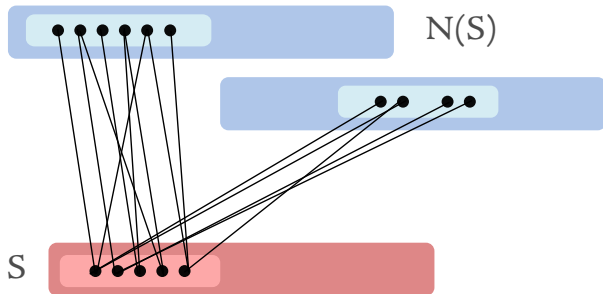


A

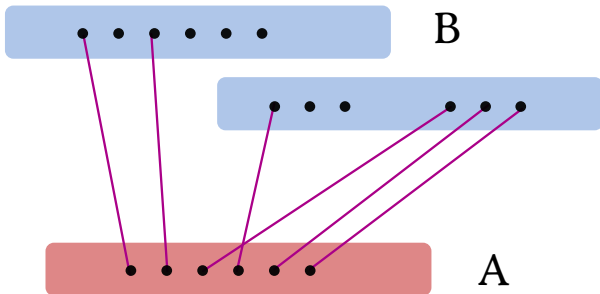
Assume that there are no isolated vertices in B.



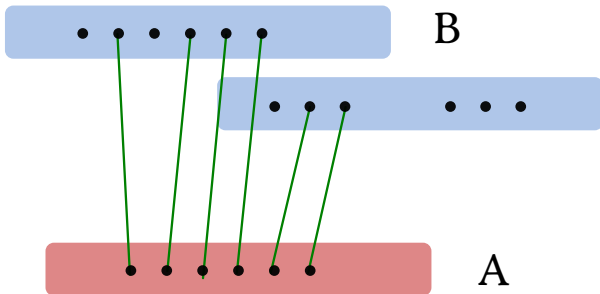
Suppose, further, that for every subset S in A ,



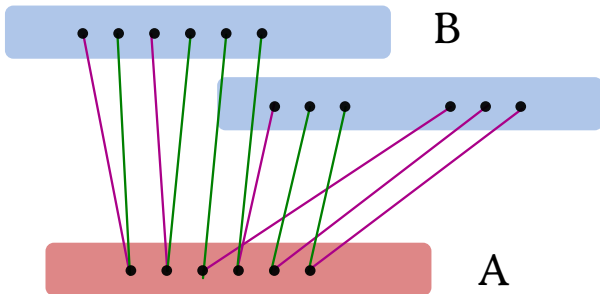
Suppose, further, that for every subset S in $\mathcal{A}$, $N(S)$ is at least twice as large as $|S|$.



Then there exist two matchings saturating A ,



Then there exist two matchings saturating A ,



Then there exist two matchings saturating A ,
and disjoint in B .



B

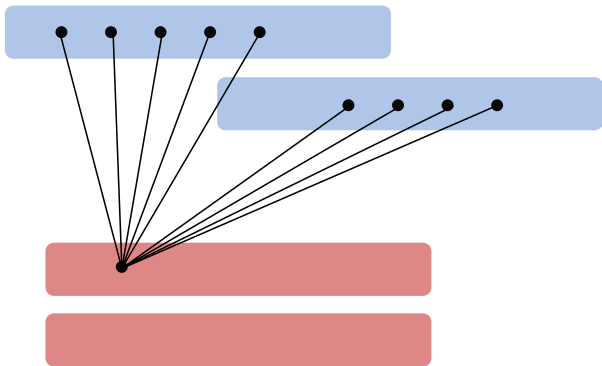


A

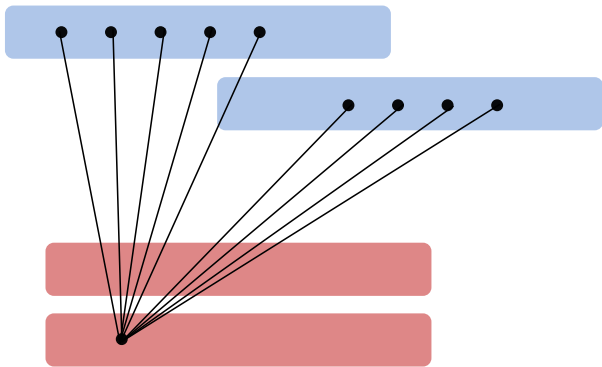
Proof?



Construct an auxiliary graph from the original by making a copy of the smaller partition.



Construct an auxiliary graph from the original by making a copy of the smaller partition.



Construct an auxiliary graph from the original by making a copy of the smaller partition.



B



A_1



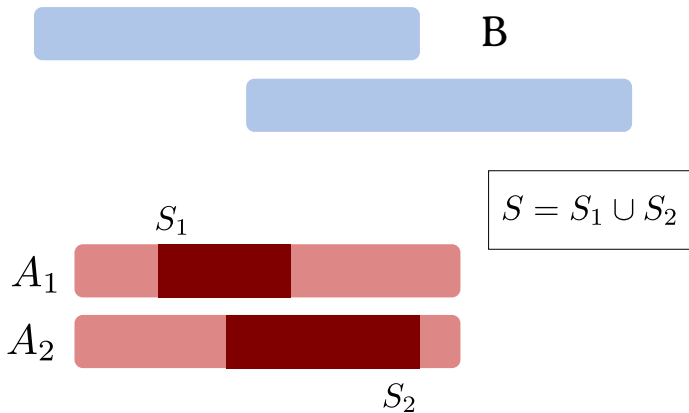
A_2



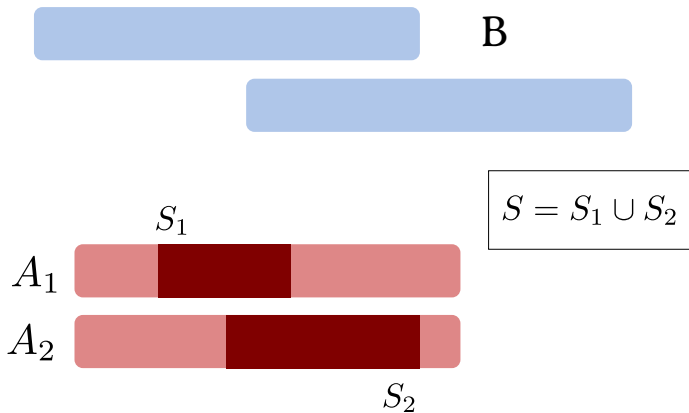
We show that Hall's original condition holds in this auxiliary graph.

To prove: For every subset S of $A_1 \cup A_2$,

$$|N(S)| \geq |S|.$$



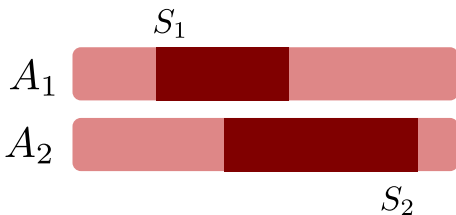
Consider any subset S of the two copies of A put together.



Let S_1 and S_2 denote $S \cap A_1$ and $S \cap A_2$, respectively.



B

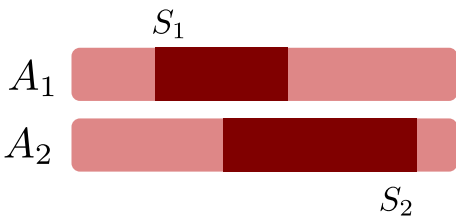


$$S = S_1 \cup S_2$$

Let $|S_2|$ be, without loss of generality,
be the larger of S_1 and S_2 .



B

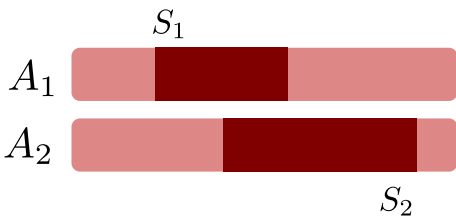


$$S = S_1 \cup S_2$$

Consider $|\mathbf{N}(S)| \geq \max\{|\mathbf{N}(S_1)|, |\mathbf{N}(S_2)|\}$



B

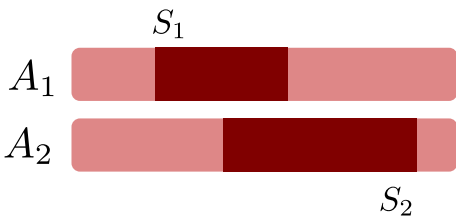


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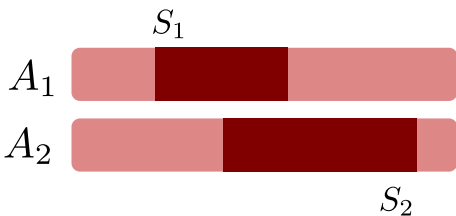


$$S = S_1 \cup S_2$$

Consider $|\mathbf{N}(S)| \geq |\mathbf{N}(S_2)|$

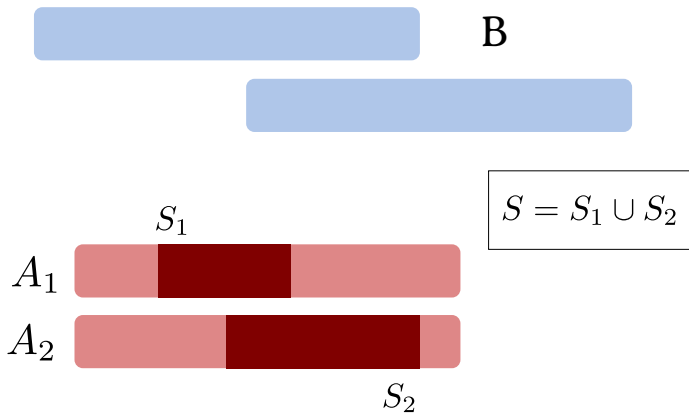


B

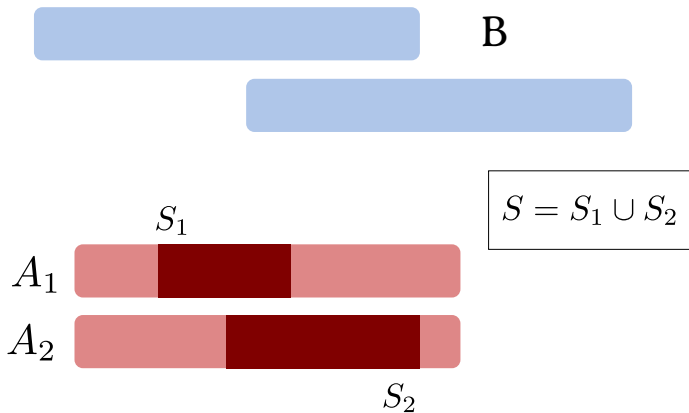


$$S = S_1 \cup S_2$$

Consider $|\mathbf{N}(S)| \geq |\mathbf{N}(S_2)|$



Consider $|\mathbf{N}(S)| \geq 2|S_2|$
 (by the assumption that $|\mathbf{N}(S_2)| \geq 2|S_2|$.)



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B

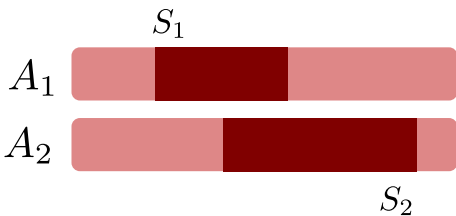


$$S = S_1 \cup S_2$$

Consider $|\mathbf{N}(S)| \geq |S_1| + |S_2|$

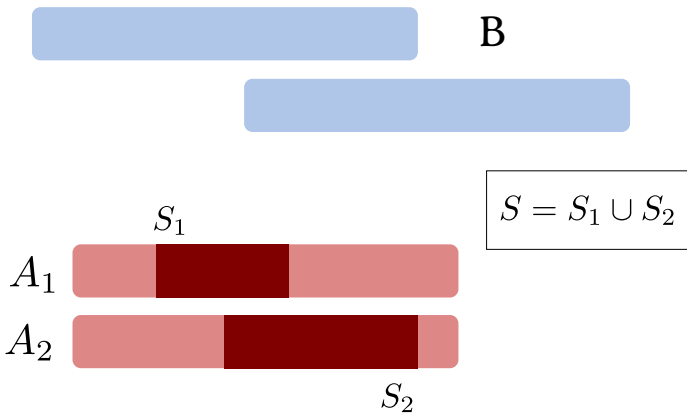


B



$$S = S_1 \cup S_2$$

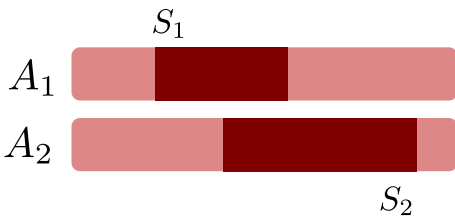
Consider $|N(S)| \geq |S_1| + |S_2|$



Consider $|N(S)| \geq |S_1| + |S_2|$
 (by the assumption that $|S_2| \geq |S_1|$.)



B



$$S = S_1 \cup S_2$$

Consider $|\mathbf{N}(S)| \geq |\mathbf{S}_1| + |\mathbf{S}_2|$



B



S_1

A_1



A_2



S_2

$$S = S_1 \cup S_2$$

$$|\mathbf{N}(S)| \geq |S|$$

So far...

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If $A \uplus B$ is a bipartite graph,

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where every subset S of A sees at least $2|S|$ vertices in B ,

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If $A \uplus B$ is a bipartite graph,

where every subset S of A sees at least $2|S|$ vertices in B ,

then there are 2 matchings saturating A that are vertex-disjoint in B .

Clearly, also:

If $A \uplus B$ is a bipartite graph,

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Clearly, also:

If $A \uplus B$ is a bipartite graph,

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Clearly, also:

If $A \uplus B$ is a bipartite graph,

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then there are q matchings saturating A that are vertex-disjoint in B .

Claim:

If $N(A) \geq q|A|$, then there exists a subset S of A such that:

there q matchings saturating the subset S
that are vertex-disjoint in B .

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If $N(A) \geq q|A|$, then there exists a subset S of A such that:

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Claim:

If $|B| \geq q|A|$, then there exists a subset S of A such that:

there q matchings saturating the subset S
that are vertex-disjoint in B ,

provided B does not have any isolated vertices.

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings,
vertex-disjoint in B ,
saturating A .

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

In this case, $S = A$ and we are done.

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$.

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. Throw it away.

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. **Throw it away.**

Delete the vertices of S and the edges incident on it,

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. **Throw it away.**

and remove any isolated vertices created in B .

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. Throw it away.

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. Throw it away. In the graph that is left, $|B|$ is still larger than $q|A|$.

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. Throw it away. In the graph that is left, $|B|$ is still larger than $q|A|$.

$$N(S) \leq q|S|$$

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

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$$- N(S) \geq - q|S|$$

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. Throw it away. In the graph that is left, $|B|$ is still larger than $q|A|$.

$$|B| - N(S) \geq q|A| - q|S|$$

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. Throw it away. In the graph that is left, $|B|$ is still larger than $q|A|$.

$$|B| - N(S) \geq q|A| - q|S|$$

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

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$$|B| - N(S) \geq q(|A| - |S|)$$

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. Throw it away.

Done: by repetition.

$$|B| - N(S) \geq q(|A| - |S|)$$

If $N(S) \geq q|S|$ for all subsets S , then there are q matchings, vertex-disjoint in B , saturating A .

Else: there exists a subset S for which $N(S)$ is smaller than $q|S|$. Throw it away.

Done: by repetition.

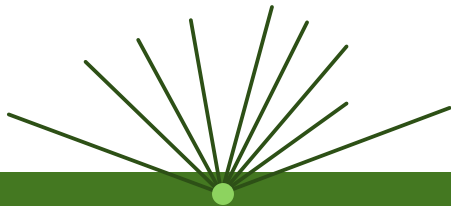
(And you'll never clear everything out, because $|B|$ is consistently more than $q|A|$.)

$$|B| - N(S) \geq q(|A| - |S|)$$

Part 2

Two-Expansions and FVS

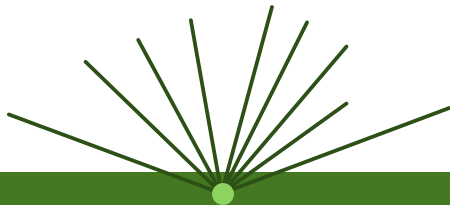
Ingredients



a high-degree vertex, v

Ingredients

a small hitting set,
sans v



a high-degree vertex, v

Ingredients

Given a high-degree vertex v , finding a small hitting set that does not contain v .

Given a high-degree vertex v , finding a small **hitting set** that does not contain v .

A subset whose removal makes the graph acyclic.

Given a high-degree vertex v , finding a **small** hitting set that does not contain v .

A polynomial function of k .

Given a **high-degree** vertex v , finding a small hitting set that does not contain v .

At least twice the size of the hitting set.

Given a high-degree vertex v , finding a small hitting set that does not contain v .

Find an approximate hitting set S .

Given a high-degree vertex v , finding a small hitting set that does not contain v .

If S does not contain v , we are done.

Given a high-degree vertex v , finding a small hitting set that does not contain v .

Else: $v \in S$. Delete $S \setminus v$ from G .

Given a high-degree vertex v , finding a small hitting set that does not contain v .

The only remaining cycles pass through v .

Given a high-degree vertex v , finding a small hitting set that does not contain v .

Find an optimal cut set for paths from $N(v)$ to $N(v)$.

Given a high-degree vertex v , finding a small hitting set that does not contain v .

Why is this cut set small enough?

Given a high-degree vertex v , finding a small hitting set that does not contain v .

When is this cut set small enough?

Given a high-degree vertex v , finding a small hitting set that does not contain v .

When is this cut set small enough?

When the largest collection of vertex disjoint paths from $N(v)$ to $N(v)$ is small.

Given a high-degree vertex v , **finding** a small hitting set that does not contain v .

When is this cut set small enough?

When the largest collection of vertex disjoint paths from $N(v)$ to $N(v)$ is **not** small...

Given a high-degree vertex v , finding a small hitting set that does not contain v .

When is this cut set small enough?

When the largest collection of vertex disjoint paths from $N(v)$ to $N(v)$ is *not* small... we get a reduction rule.

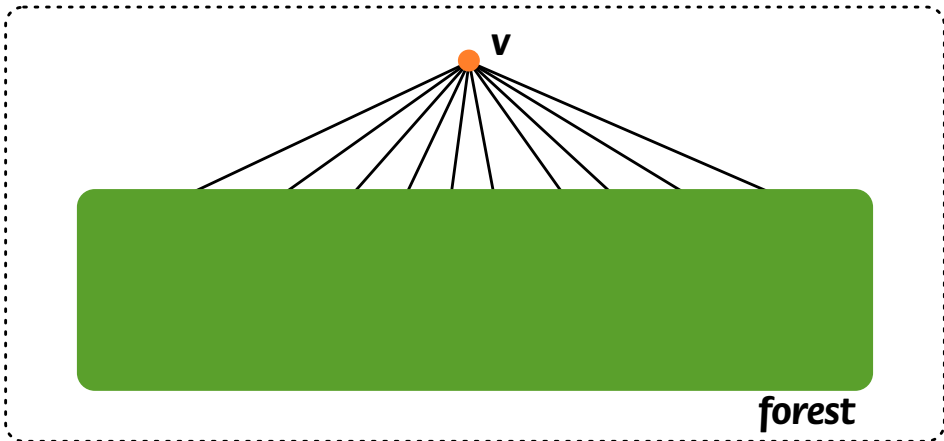
Given a high-degree vertex v , finding a small hitting set that does not contain v .

When is this cut set small enough?

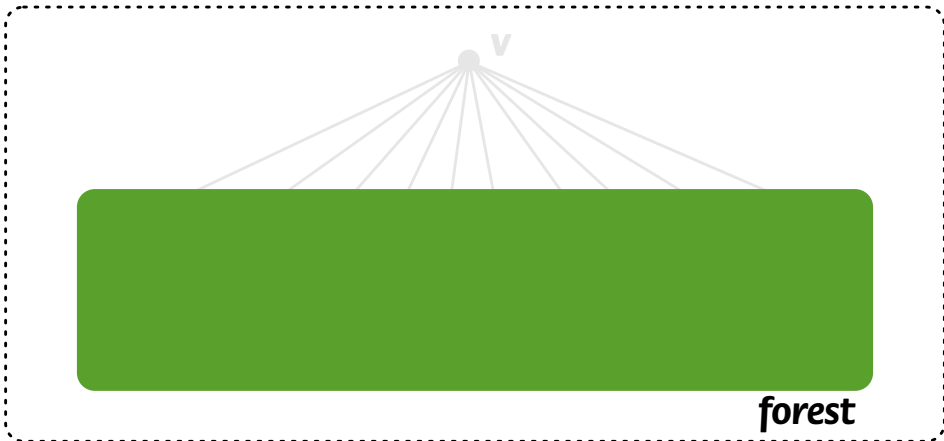
More than k vertex-disjoint paths from $N(v)$ to $N(v)$
→ v belongs to *any* hitting set of size at most k .

Given a high-degree vertex v , finding a small hitting set that does not contain v .

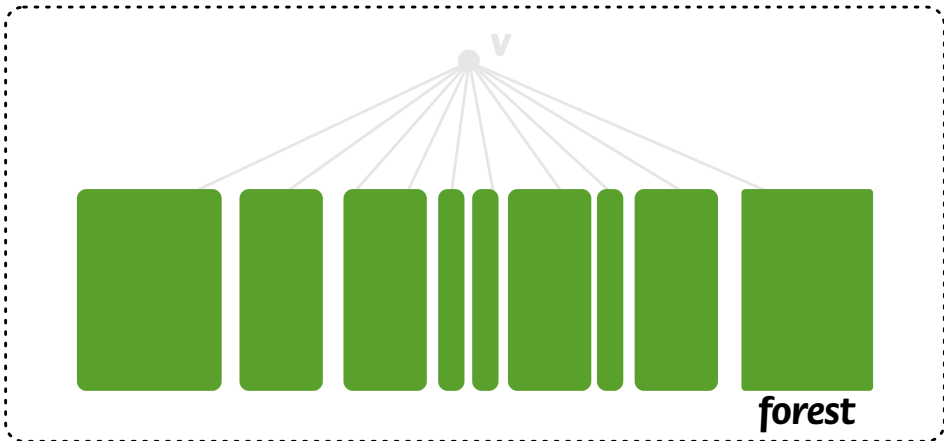
So either v “forced”, or we have hitting set of suitable size.
Notice that we arrive at either situation in polynomial time.



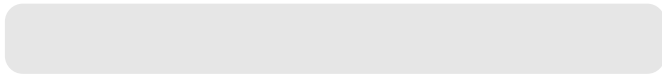
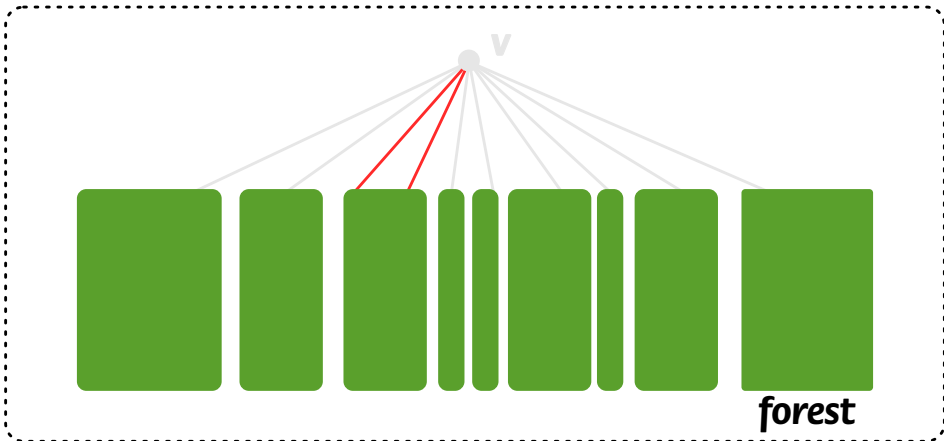
hitting set that excludes v



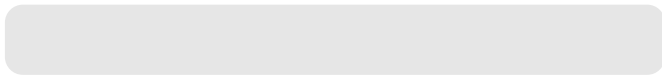
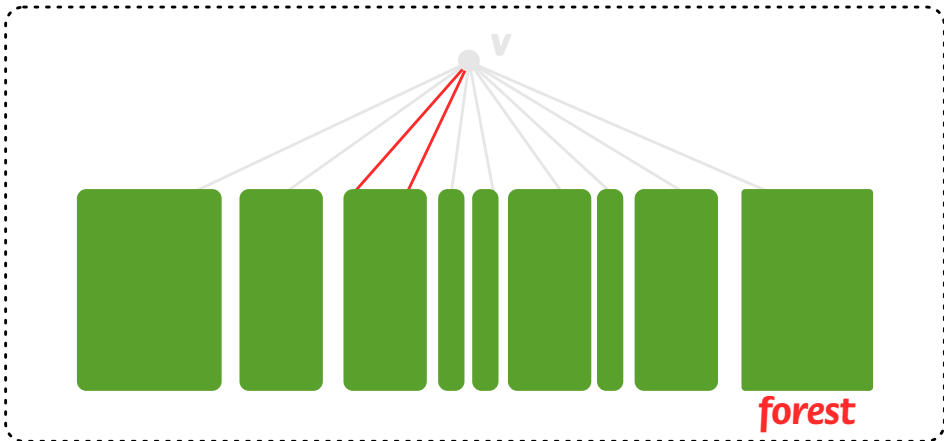
hitting set that excludes v



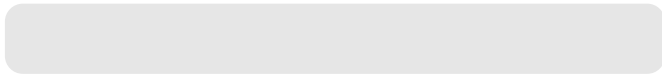
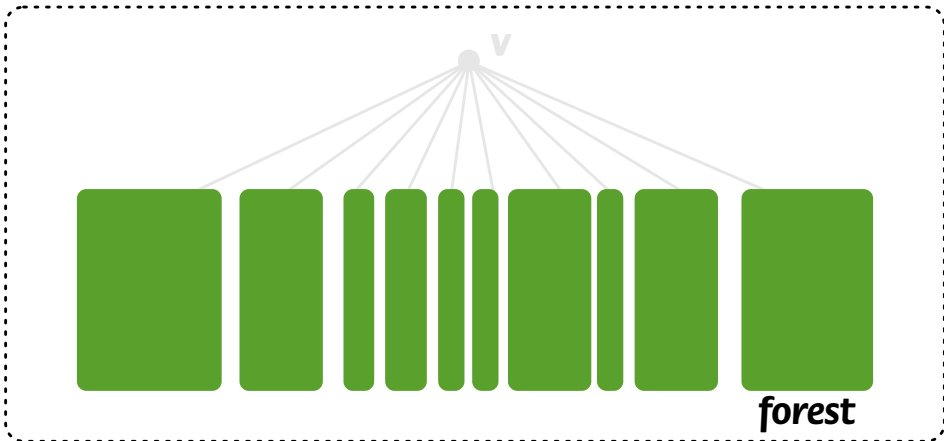
hitting set that excludes v



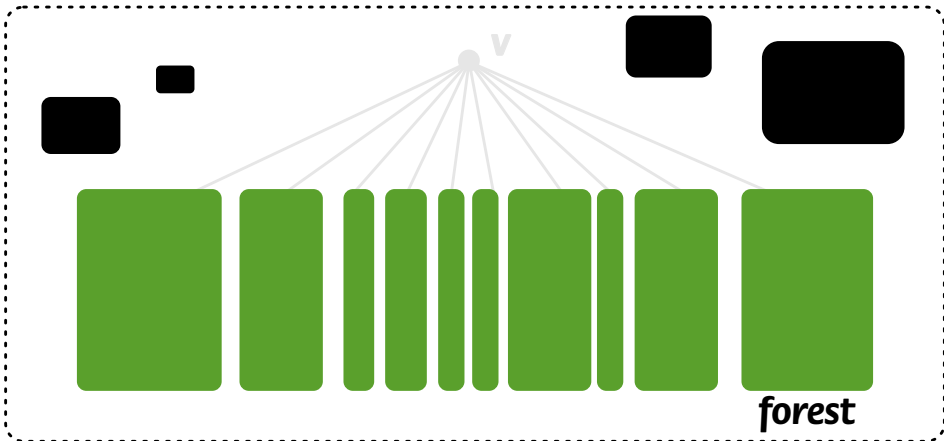
hitting set that excludes v



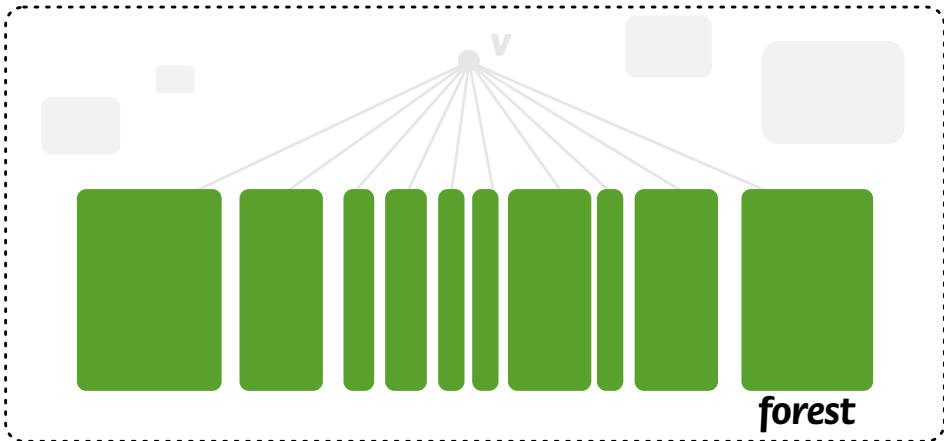
hitting set that excludes v



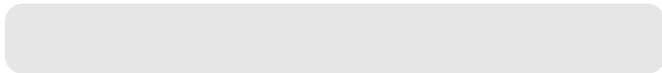
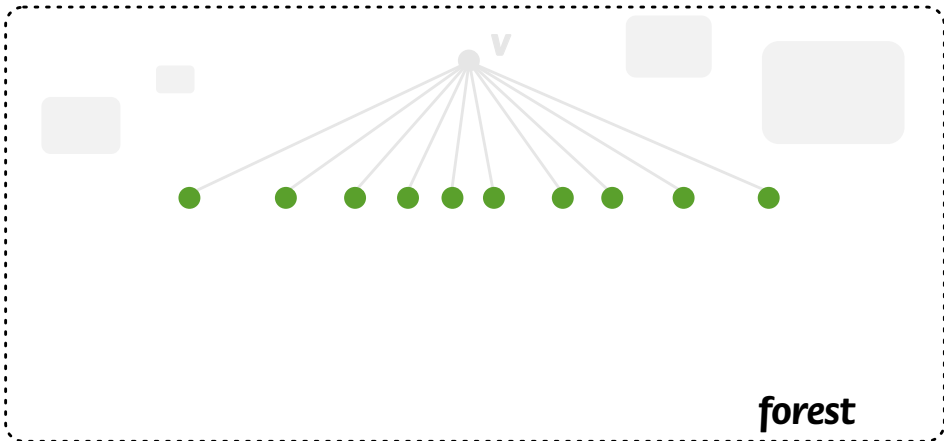
hitting set that excludes v



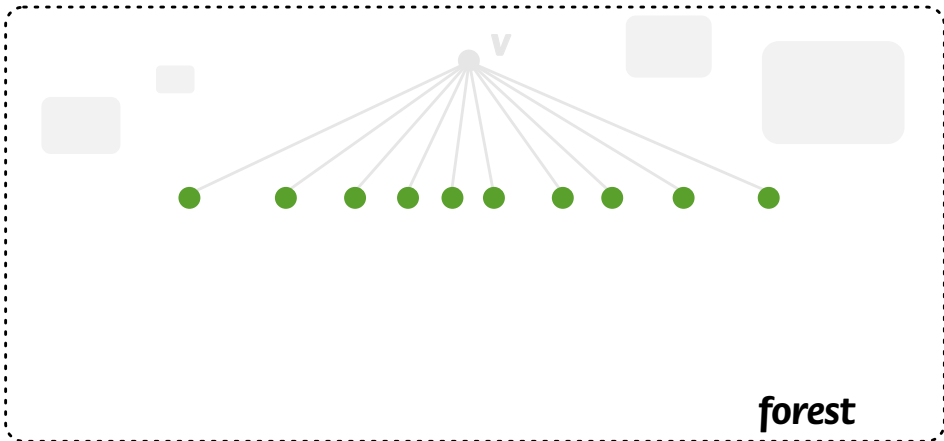
hitting set that excludes v



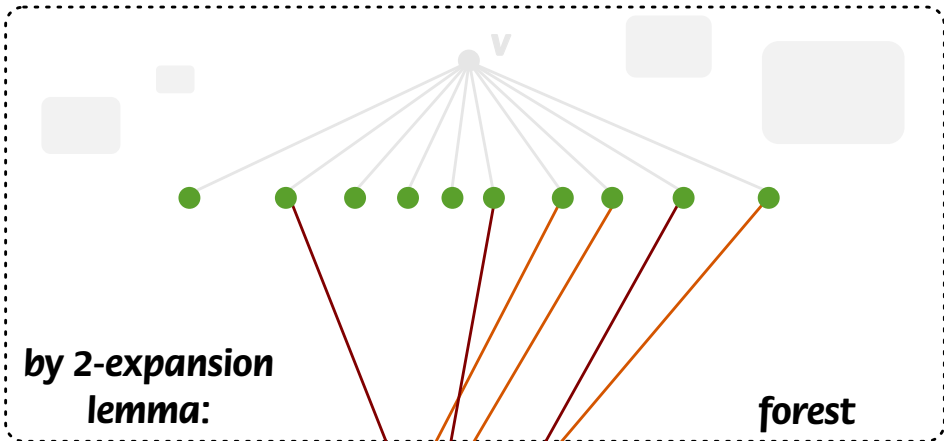
hitting set that excludes v



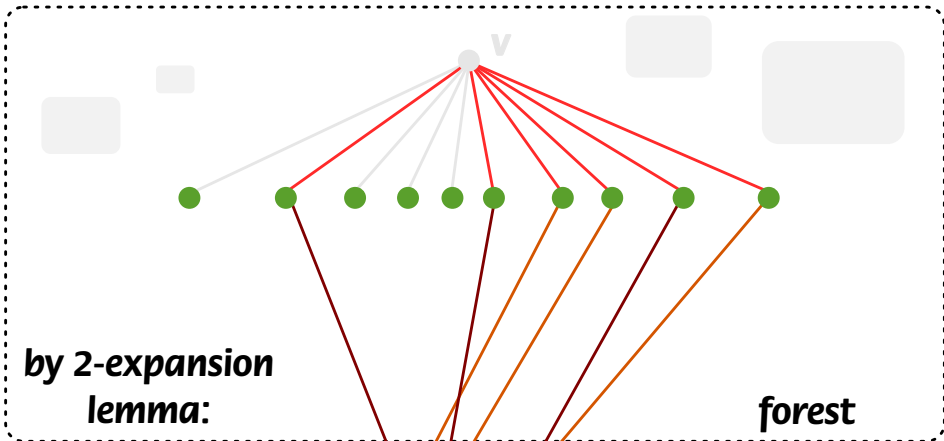
hitting set that excludes v



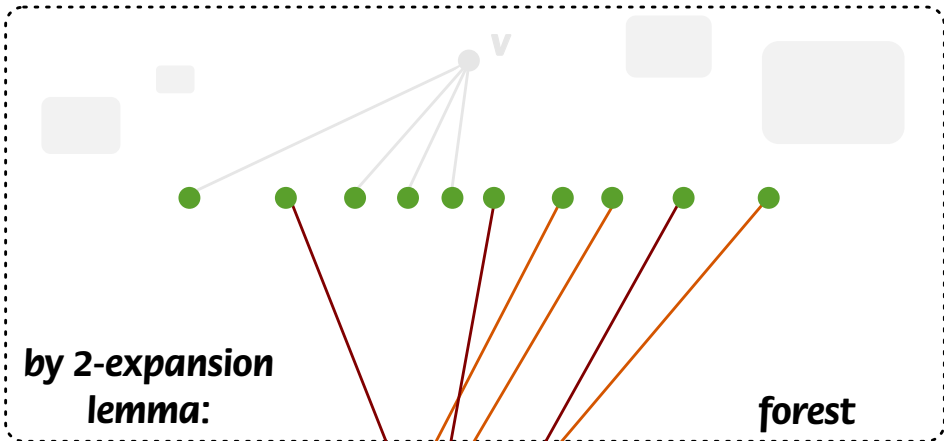
hitting set that excludes v



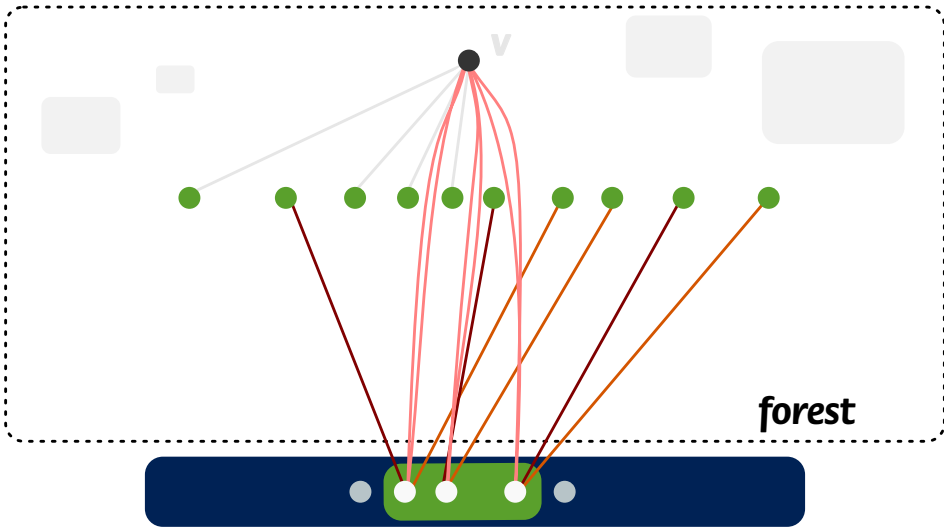
hitting set that excludes v



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hitting set that excludes v

The Forward Direction

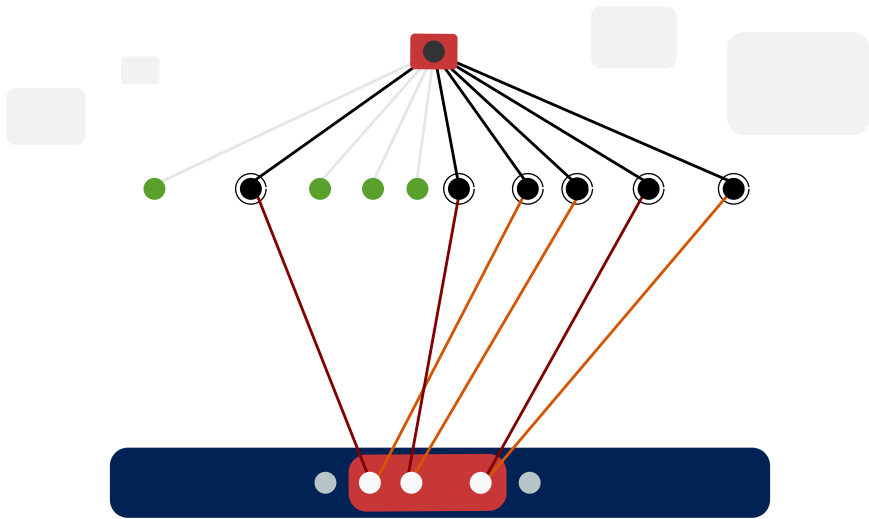
The Forward Direction

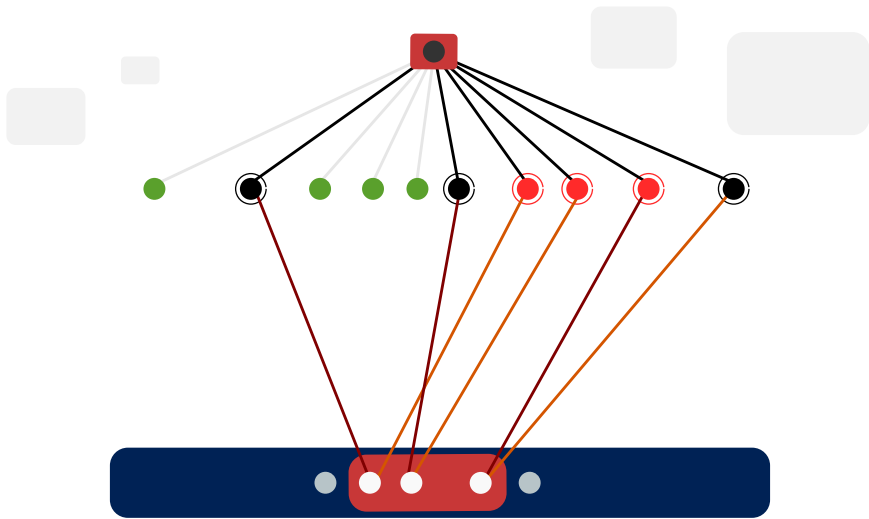
$$\text{FVS} \leq k \text{ in } G \Rightarrow \text{FVS} \leq k \text{ in } H$$

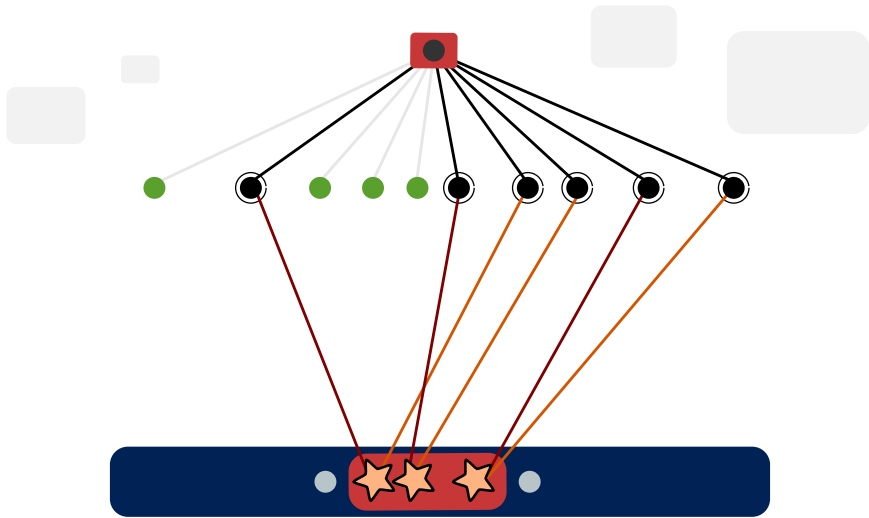
If G has a FVS that either contains v or all of S , we are in good shape.

Consider now a FVS that:

- ✦ Does not contain v ,
- ✦ and omits at least one vertex of S .







Notice that this does not lead to a larger FVS:

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For every vertex v in S that a FVS of G leaves out,

Notice that this does not lead to a larger FVS:

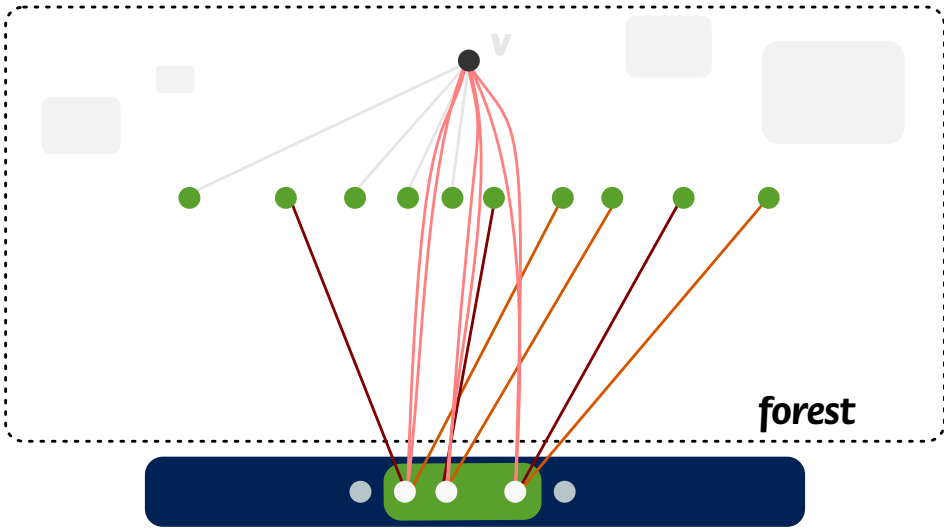
For every vertex v in S that a FVS of G leaves out,

it must pick a vertex u that kills no more than all of S .

The Reverse Direction

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$$\text{FVS} \leq k \text{ in } G \Leftarrow \text{FVS} \leq k \text{ in } H$$



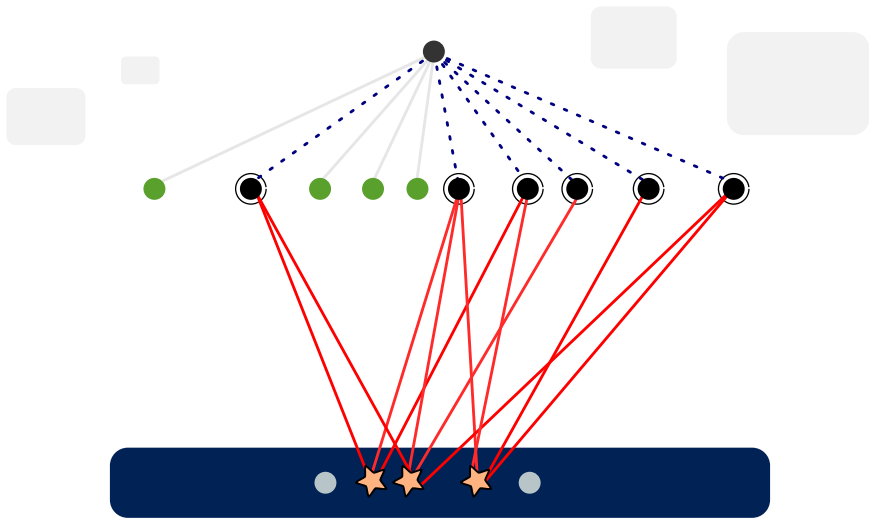
hitting set that excludes v

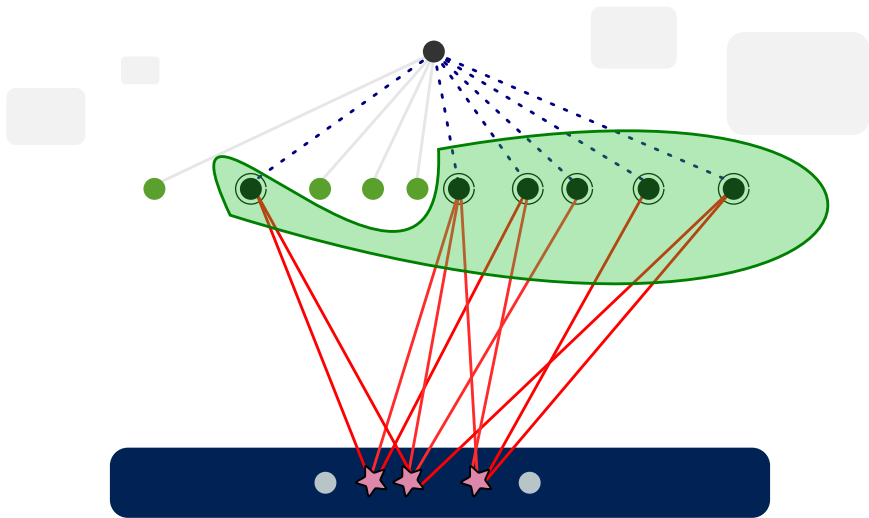
The Only Danger

Cycles that:

- ♦ pass through v ,
- ♦ non-neighbors of v in H^1
- ♦ and do not pass through S .

¹neighbors in G , however

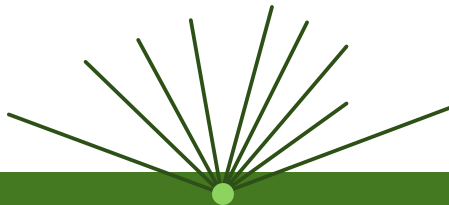




Part q

*q-Expansions for Hitting
 Θ_q -minor models*

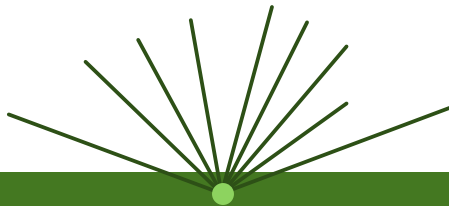
a hitting set,
sans v



a high-degree vertex, v

Ingredients

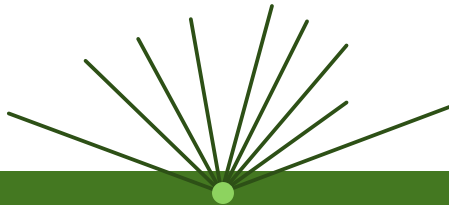
a hitting set,
sans v



a high-degree vertex, v

Ingredients

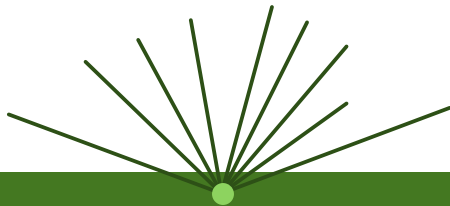
an approximate
hitting set



a high-degree vertex, v

Ingredients

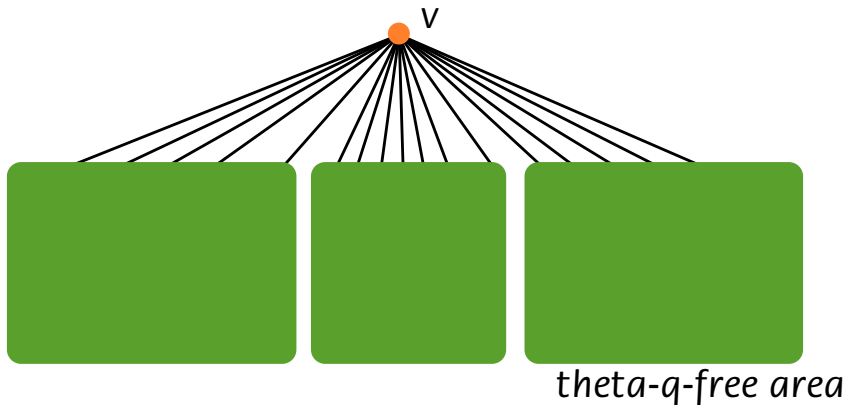
a way of detecting
($k+1$)-flowers



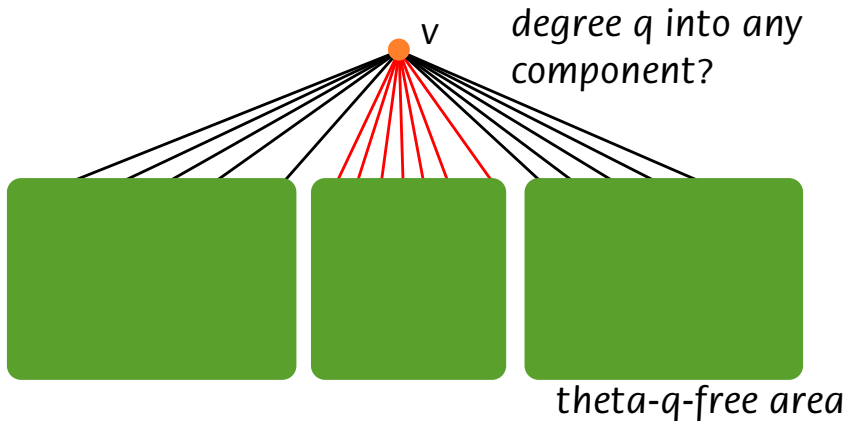
a high-degree vertex, v

Ingredients

Assuming we have a small hitting set, and a vertex of large degree...



hitting set that excludes v



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So the degree into every component is bounded by $(q - 1)$,

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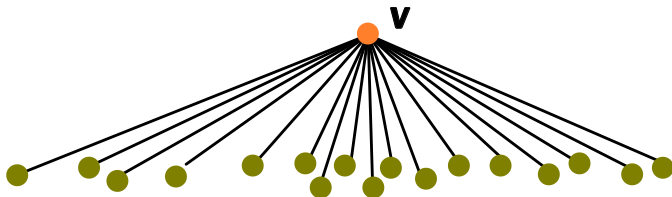
If there are at least

$$q|HS|$$

components, we get a q -expansion!

If we can handle the expansion, note that we will end up with a degree bound of:

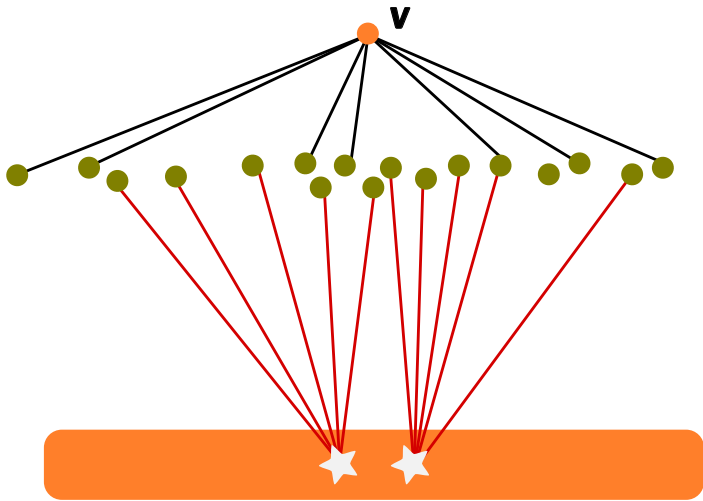
$$q(q - 1)\text{poly}(k).$$

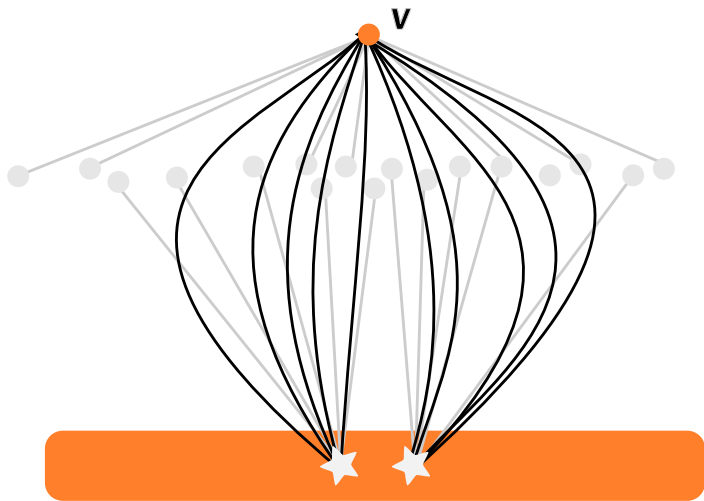


theta-q-free area



hitting set that excludes v





The Forward Direction

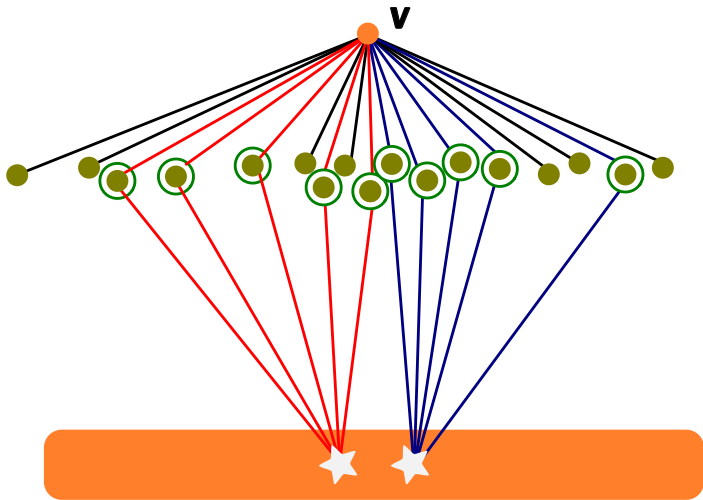
The Forward Direction

$$\text{Solution} \leq k \text{ in } G \Rightarrow \text{Solution} \leq k \text{ in } H$$

If G has a solution that either contains v or all of S , we are in good shape.

Consider now a solution that:

- ✦ Does not contain v ,
- ✦ and omits at least one vertex of S .



Notice that this does not lead to a larger solution:

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For every vertex v in S that a solution in G leaves out,

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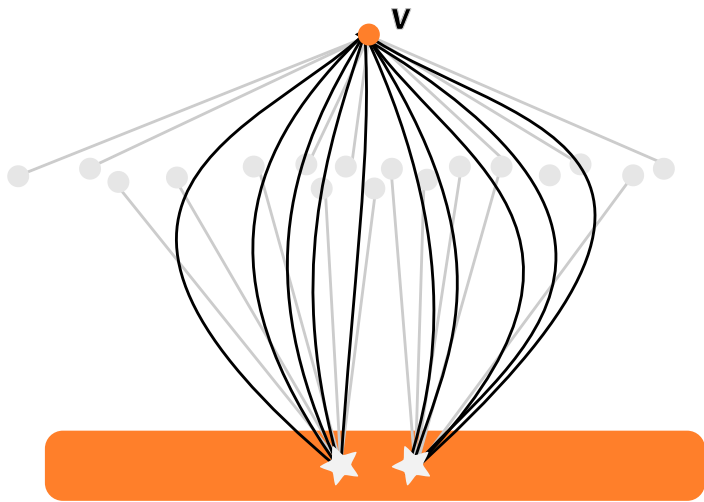
For every vertex v in S that a solution in G leaves out,

it must pick a vertex u that kills no more minor models than all of S .

The Reverse Direction

The Reverse Direction

$$\text{Solution} \leq k \text{ in } G \Leftarrow \text{Solution} \leq k \text{ in } H$$



The Only Danger

Minor models that:

- ♦ pass through v ,
- ♦ non-neighbors of v in H^2
- ♦ and do not pass through S .

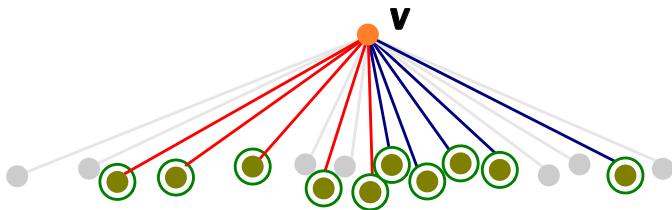
²neighbors in G , however

The Only Danger

Minor models that:

- ♦ pass through v ,
- ♦ non-neighbors of v in H^2
- ♦ and **do not pass through S .**

²neighbors in G , however



If there were minor models here, then v would be a cut-vertex. Since θ_c is a 2-connected graph: contradiction!