

# ON THE INFEASIBILITY OF OBTAINING POLYNOMIAL KERNELS

why some problems are incompressible

# PROBLEM KERNELS

how why what

when

## NO POLYNOMIAL KERNELS

### EXAMPLES

satisfiability

vertex-disjoint cycles

disjoint factors

### WORKAROUNDS

many kernels

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The professor would like to locate a group of *six* people who are popular.

i.e, everyone is a friend of at least one of the six.

Being a busy man, he asks two of his students to find such a group.

The first grimaces and starts making a list of  $\binom{25}{6}$  possibilities.

The second knows that no one in the party has more than three friends<sup>1</sup>.

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but is trivial on “large” graphs of bounded degree,

as you can say NO whenever  $n > kb$ .

Pre-processing is a humble strategy for coping with hard problems,  
almost universally employed.

Mike Fellows

We would like to talk about the “preprocessing complexity” of a problem.

To what extent may we simplify/compress a problem before we begin to solve it?

Note that any compression routine has to *run efficiently* to look attractive.

This makes it unreasonable for any NP-complete problem to admit compression algorithms.

However, (some) NP-complete problems can be compressed.

We extend our notion (and notation) to understand them.

## Notation

We denote a parameterized problem as a pair  $(Q, \kappa)$  consisting of a classical problem  $Q \subseteq \{0, 1\}^*$  and a parameterization  $\kappa : \{0, 1\}^* \rightarrow \mathbb{N}$ .

Data reduction

involves pruning down

a large input

into an equivalent,

significantly smaller object,

quickly.

Data reduction

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Size of the Kernel

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## Theorem

Having a kernelization procedure implies, and is implied by,  
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## Definition

A parameterized problem  $L$  is *fixed-parameter tractable* if there exists an algorithm that decides in  $f(k) \cdot n^{O(1)}$  time whether  $(x, k) \in L$ , where  $n := |x|$ ,  $k := \kappa(x)$ , and  $f$  is a computable function that does not depend on  $n$ .

## Theorem

A problem admits a kernel if, and only if,  
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In particular,

Polynomial-sized kernels       $\prec$ <sub>are better than</sub>      Exponential-sized Kernels

The problem of finding Dominating Set of size  $k$  on graphs where the degree is bounded by  $b$ , *parameterized by*  $k$ , has a linear kernel. This is an example of a polynomial-sized kernel.

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A composition algorithm  $\mathcal{A}$  for a problem is designed to act as a fast Boolean OR of multiple problem-instances.

It receives as input a sequence of instances.

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$\bar{x} = (x_1, \dots, x_t)$  with  $x_i \in \{0, 1\}^*$  for  $i \in [t]$ , such that  
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Running time polynomial in  $\sum_{i \in [t]} |x_i|$

## The Recipe for Hardness

Composition Algorithm + Polynomial Kernel



Distillation Algorithm



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$$\text{PH} = \Sigma_3^{\text{P}}$$

Theorem. Let  $(P, k)$  be a compositional parameterized problem such that  $P$  is NP-complete. If  $P$  has a polynomial kernel, then  $P$  also has a distillation algorithm.

## Transformations

Let  $(P, \kappa)$  and  $(Q, \gamma)$  be parameterized problems.

We say that there is a polynomial parameter transformation from  $P$  to  $Q$  if there exists a polynomial time computable function  $f : \{0, 1\}^* \longrightarrow \{0, 1\}^*$ , and a polynomial  $p : \mathbb{N} \rightarrow \mathbb{N}$ , such that, if  $f(x) = y$ , we have:

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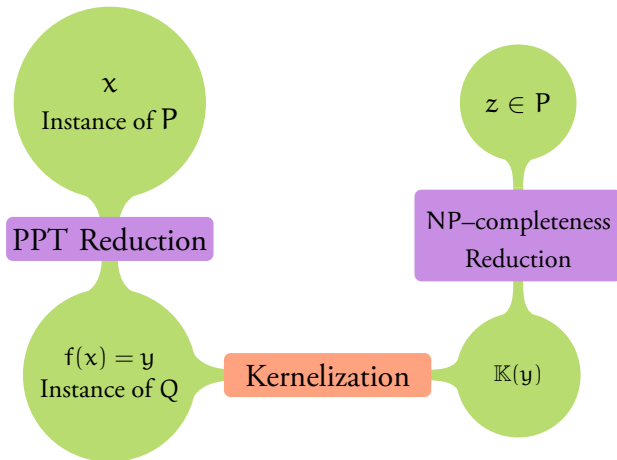
$x \in P$  if and only if  $y \in Q$ ,

and

$$\gamma(y) \leq p(\kappa(x))$$

Theorem: Suppose  $P$  is NP-complete, and  $Q \in \text{NP}$ . If  $f$  is a polynomial time and parameter transformation from  $P$  to  $Q$  and  $Q$  has a polynomial kernel, then  $P$  has a polynomial kernel.

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## Recall

A composition for a parameterized language  $(Q, \kappa)$  is required to “merge” instances

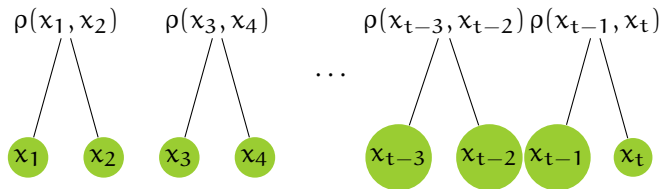
$$x_1, x_2, \dots, x_t,$$

into a single instance  $x$  in polynomial time, such that  $\kappa(x)$  is polynomial in  $k := \kappa(x_i)$  for any  $i$ .

The output of the algorithm belongs to  $(Q, \kappa(x))$   
if, and only if  
there exists at least one  $i \in [t]$  for which  $x_i \in (Q, \kappa)$ .

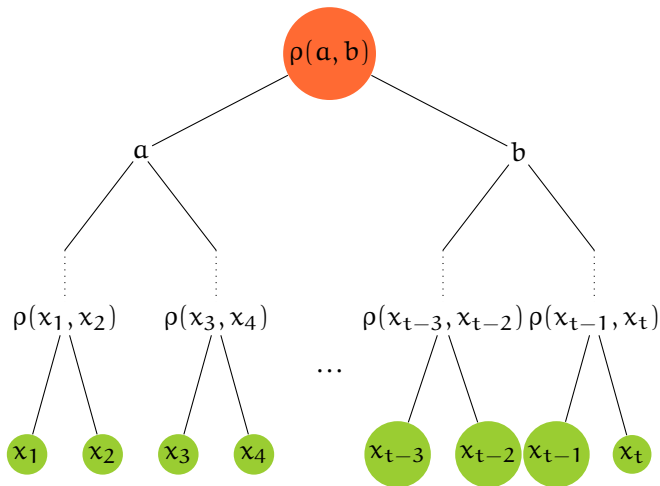


General Framework For Composition



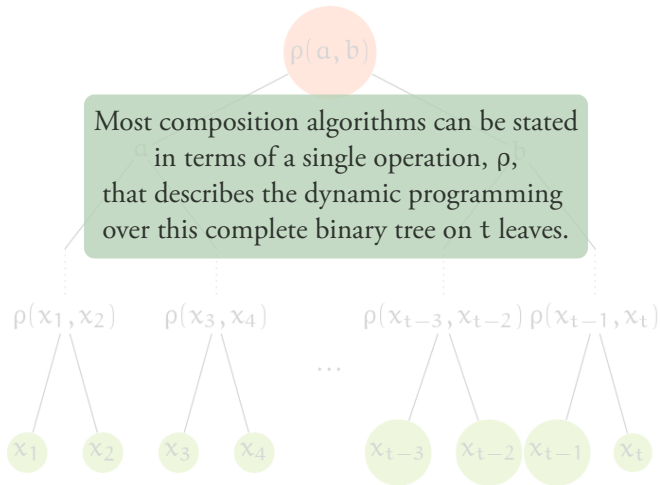
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Parameter:  $b+k$ .

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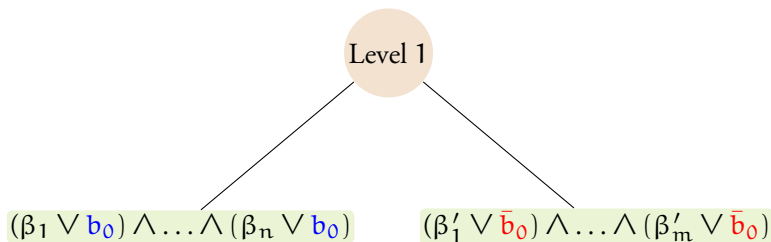
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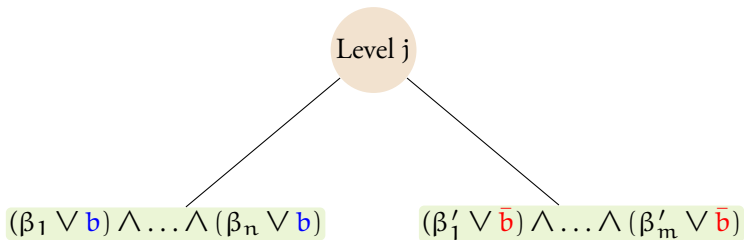
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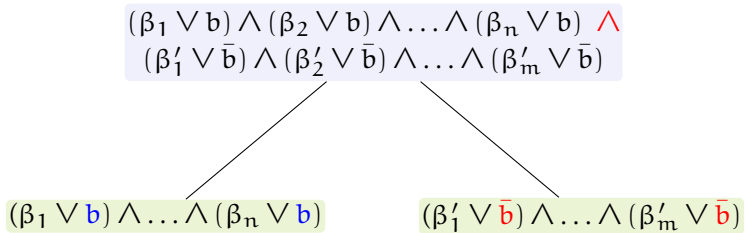
Total time  $< t \cdot t \cdot p(n)$

If not,  $t < b^k$  – this gives us a bound on the number of instances.



This is the scene at the leaves, where the  $\beta_j$ s are the clauses in  $\alpha_i$  for some  $i$  and the  $\beta'_j$ s are clauses of  $\alpha_{i+1}$ .





Take the conjunction of the formulas stored at the child nodes.

$$(\beta_1 \vee b \vee b_j) \wedge (\beta_2 \vee b \vee b_j) \wedge \dots \wedge (\beta_n \vee b \vee b_j) \wedge$$

$$(\beta'_1 \vee \bar{b} \vee b_j) \wedge (\beta'_2 \vee \bar{b} \vee b_j) \wedge \dots \wedge (\beta'_m \vee \bar{b} \vee b_j)$$

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if the parent is a “left child”.

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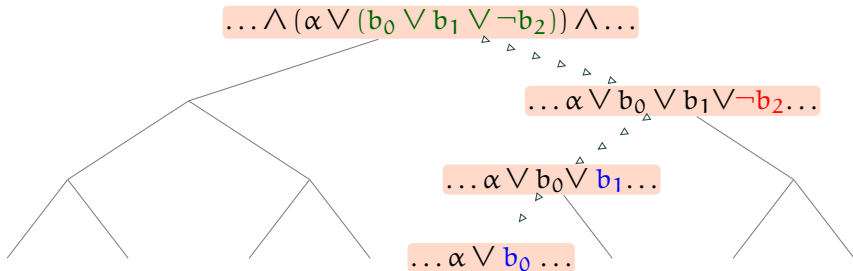
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if the parent is a “right child”.

## Proof of Correctness



# Disjoint Factors

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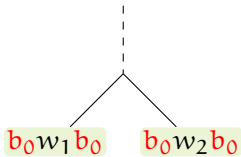
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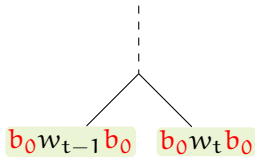
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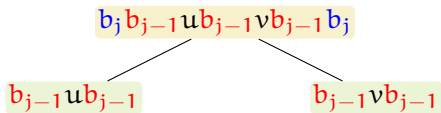
## Leaf Nodes



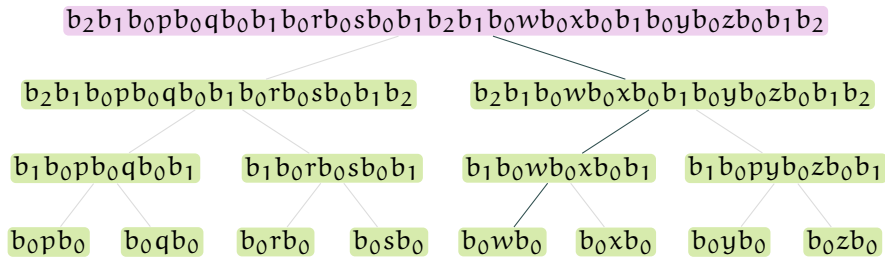
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Level  $j$ .

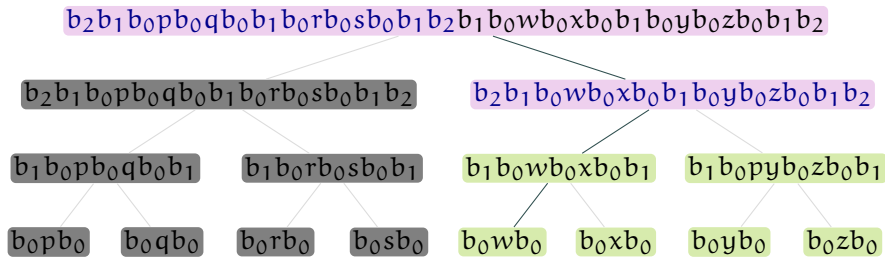


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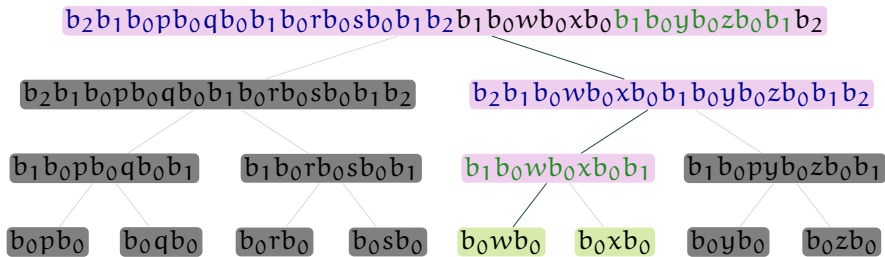
Input words:  $p, q, r, s, w, x, y, z$ .

## Proof of Correctness



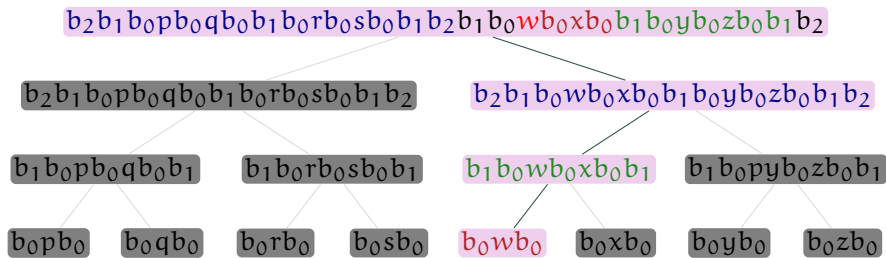
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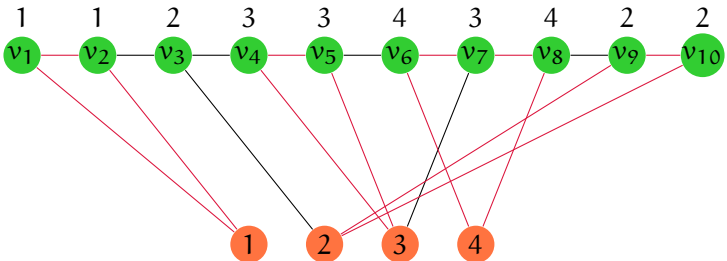
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In contrast, Disjoint Factors transforms into Vertex-Disjoint Cycles in polynomial time.

$$w = 1123343422$$



Disjoint Factors  $\preceq_{\text{ppt}}$  Disjoint Cycles

# PROBLEM KERNELS

how why what

when

## NO POLYNOMIAL KERNELS

### EXAMPLES

satisfiability

vertex-disjoint cycles

disjoint factors

## WORKAROUNDS

many kernels

No polynomial kernel?

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No polynomial kernel?

Look for the best exponential or subexponential kernels...

... or build *many* polynomial kernels.

Many polynomial kernels give us better algorithms:

$$\binom{k^2}{k}$$

$$2^{k \log k}$$

versus

$$p(n)c^k$$

$$p(n) \cdot \binom{ck}{k}$$

We say that a subdigraph  $T$  of a digraph  $D$  is an *out-tree* if  $T$  is an oriented tree with only one vertex  $r$  of in-degree zero (called the *root*).

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However, it clearly admits  $n$  polynomial kernels (try all possible choices for root, and apply the kernel for **ROOTED**  $k$ -LEAF OUT-BRANCHING).

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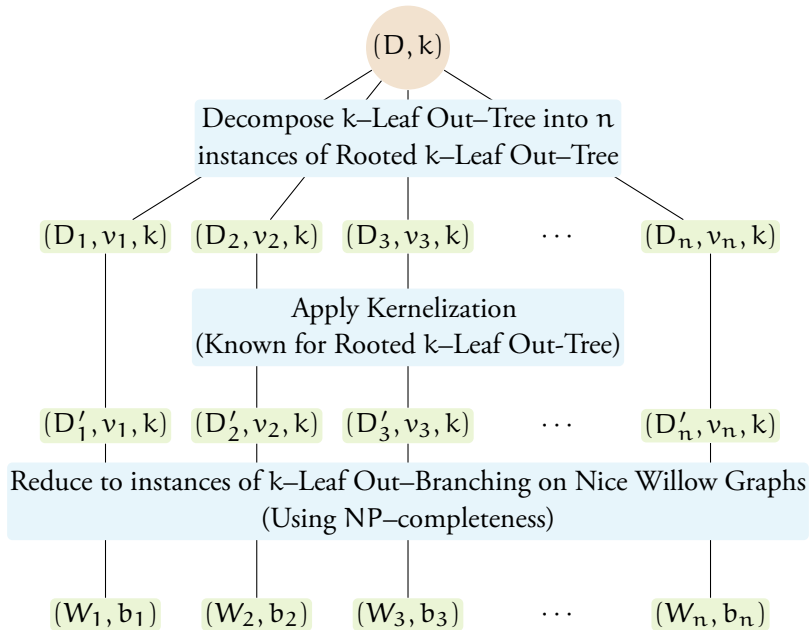
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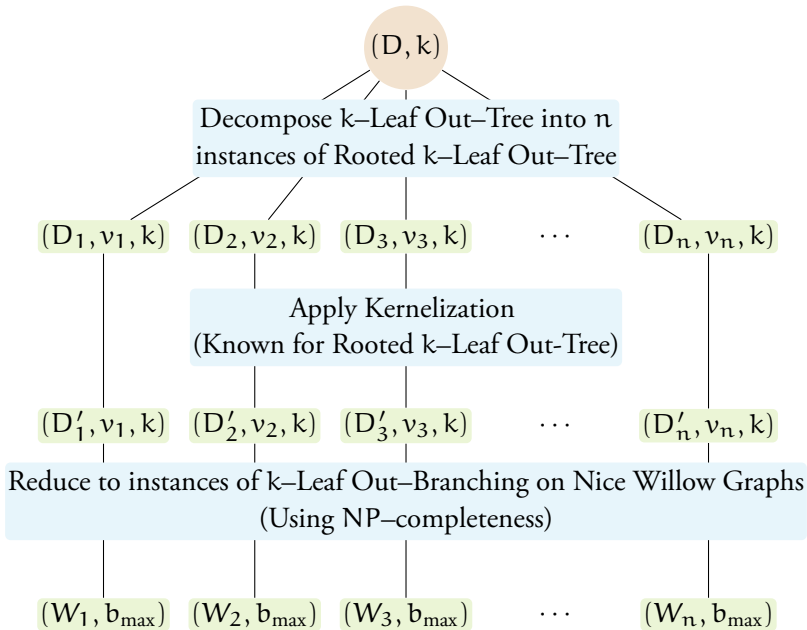
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Composition  
(produces an instance of  $k$ -Leaf Out Branching)

$(D', b_{\max} + 1)$

Kernelization  
(Given By Assumption)

$(D'', k'')$

NP-Completeness reduction from  $k$ -Leaf Out Branching  
to  $k$ -Leaf Out Tree)

$(D^*, k^*)$

A polynomial pseudo-kernelization  $K$  produces kernels whose size can be bounded by:

$$h(k) \cdot n^{1-\varepsilon}$$

where  $k$  is the parameter of the problem.

CONCLUDING  
REMARKS

Analogous to the composition framework, there are algorithms (called **Linear OR**) whose existence helps us rule out polynomial pseudo-kernels.

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Most compositional algorithms can be extended to fit the definition of Linear OR.

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Mechanisms for ruling out strong kernels are simpler and rely on self-reductions and the hypothesis that  $P \neq NP$ .

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A **strong kernel** is one where the parameter of the kernelized instance is at most the parameter of the original instance.

Mechanisms for ruling out strong kernels are simpler and rely on self-reductions and the hypothesis that  $P \neq NP$ .

Example:  $k$ -Path is not likely to admit a strong kernel.

CONCLUDING  
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There are no known ways of inferring that a problem is unlikely to have a kernel of size  $k^c$  for some specific  $c$ .

*Open Problems*

Can lower bound theorems be proved under some “well-believed” conjectures of parameterized complexity - for instance -  $\text{FPT} \neq \text{XP}$ , or,  $\text{FPT} \neq \text{W}[t]$  for some  $t \in \mathbb{N}^+$ ?

*Open Problems*

A lower bound framework for ruling out  $p(k) \cdot f(l)$ -sized kernels for problems with two parameters  $(k, l)$  would be useful.

*Open Problems*

“Many polynomial kernels” have been found only for the directed outbranching problem. It would be interesting to apply the technique to other problems that are not expected to have polynomial-sized kernels.

*Open Problems*

FIN