

PC301. Iterative Compression

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graph TD; A[PC301. Iterative Compression] --> B[Warm up with Vertex Cover]; A --> C[An illustration with Feedback Vertex Set];
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Warm up with
Vertex Cover

An illustration with
Feedback Vertex Set

Vertex Cover

Input: A graph G & a budget k

Question: $\exists S \subseteq V$ such that $G \setminus S$ is an independent set
& $|S| \leq k$?

Parameter: k

Vertex Cover

S^* *does this help?*
& a vertex cover of size $\leq k+1$

Input: A graph G & a budget k) 

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Quick sanity check

is S^* minimal?

if $\exists v \in S^*$ s.t. $S^* \setminus \{v\}$ is a VC ✓

o/w, no proper subset of S^* is a vertex cover.

Vertex Cover

S^* & a vertex cover of size $\leq k+1$

does this help?

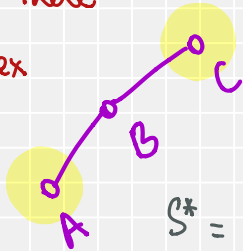
Input: A graph G & a budget k



Question: $\exists S \subseteq V$ such that $G \setminus S$ is an independent set
& $|S| \leq k$?

Note that the non-minimality of S^* does not imply that there is no vertex cover of size $\leq k$.

Parameter: k



$S^* = \{A, C\}$

Quick sanity check

is S^* minimal?

if $\exists v \in S^*$ s.t. $S^* \setminus \{v\}$ is a VC ✓

o/w, no proper subset of S^* is a vertex cover.

Vertex Cover

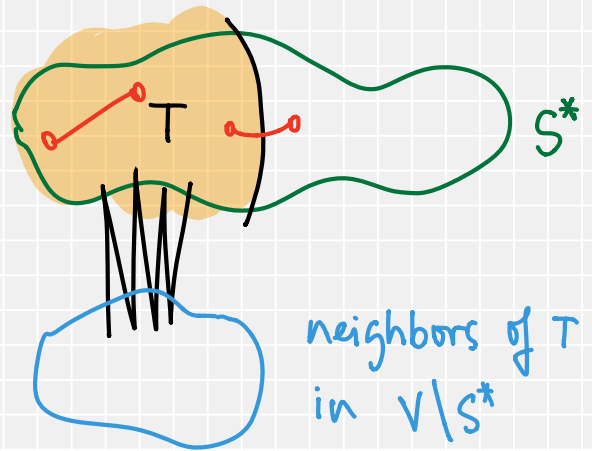
& a vertex cover of size $\leq k+1$

Input: A graph G & a budget k 

Question: $\exists S \subseteq V$ such that $G \setminus S$ is an independent set
& $|S| \leq k$?

When
does an
"exchange"
make sense?

Parameter: k



Vertex Cover

& a vertex cover of size $\leq k+1$

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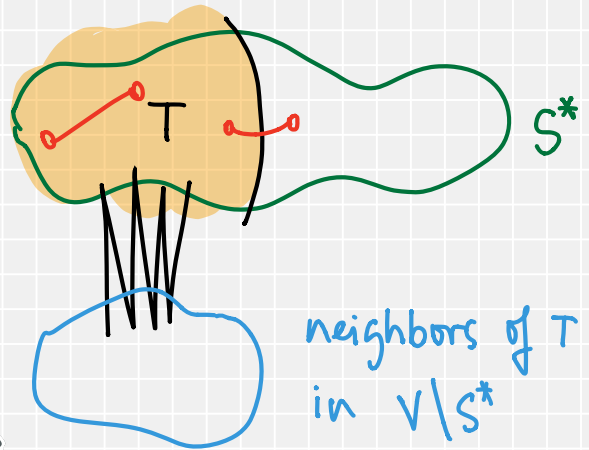
When
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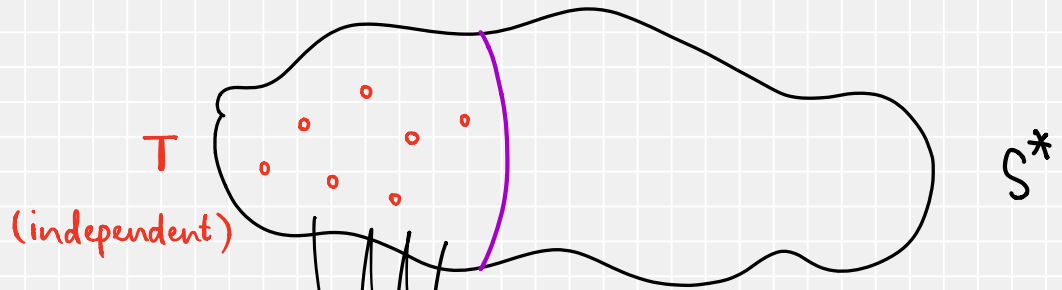
Parameter: k

when $\exists T \subseteq S$

such that $|N(T) \cap (V \setminus S^*)| < |T|$

& $G[T]$ is an independent set

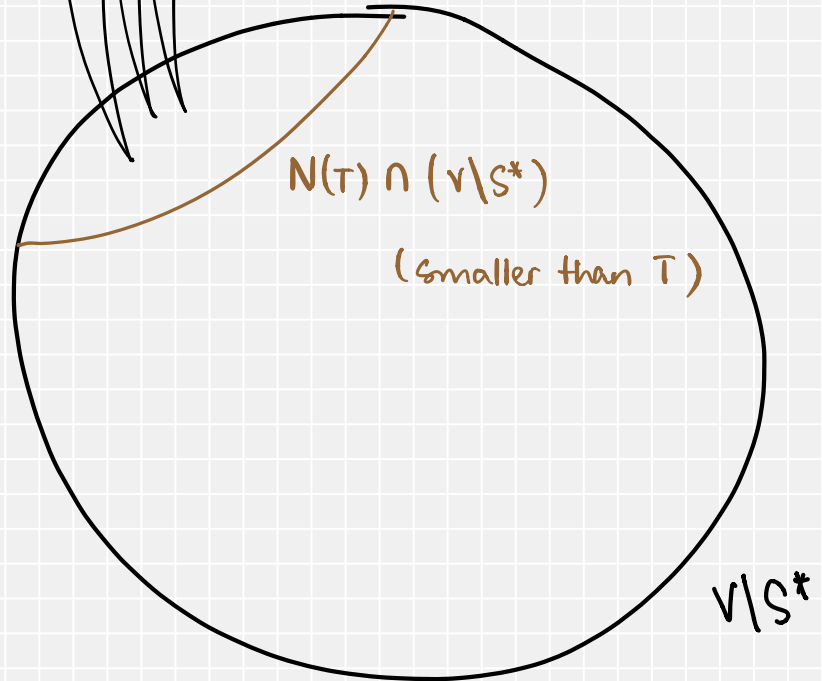


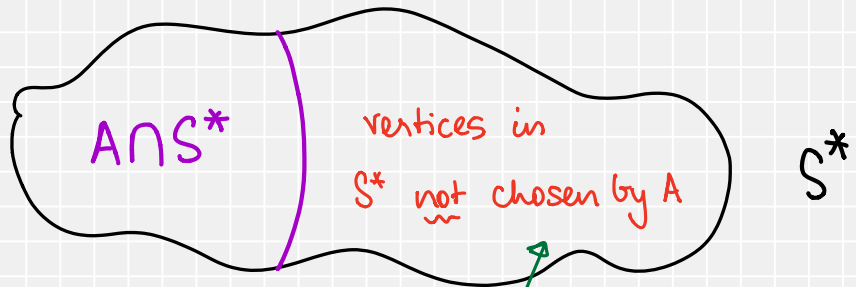


if we say YES.

$$(S^* \setminus T) \cup N(T)$$

is a vertex cover
of size $\leq k$.





if we say NO

and $\exists$ a VC of size $\leq k$

A

Contradiction

Set $T = (S^* - (S^* \cap A))$

for this choice
of T in our exhaustive
search, we would
have discovered A .

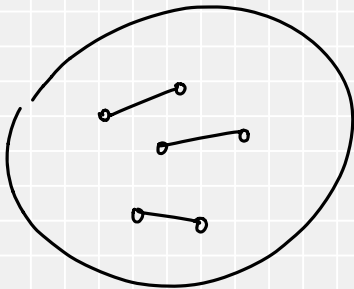
$V \setminus S^*$

Getting rid of



Method 1. Use an approximate solution.

(Eg, a 2-approximate vertex cover
given by both endpoints of
a maximal matching)

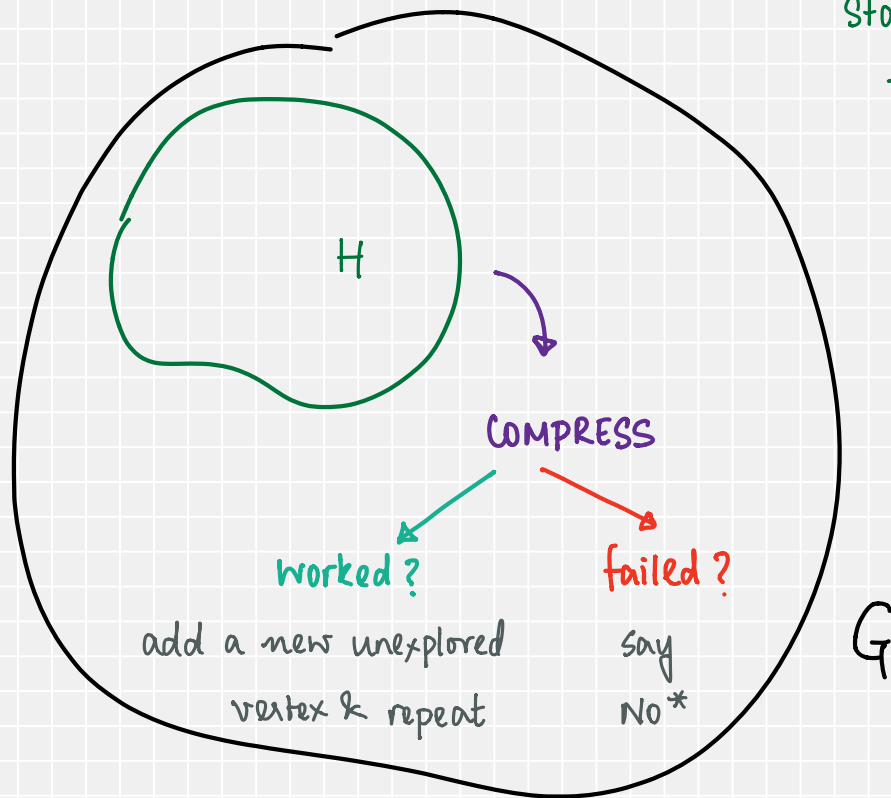


maximal
matching

$> k$ edges? Say NO

$\leq k$ edges? $2k$ -sized VC.

Method 2. Iterate

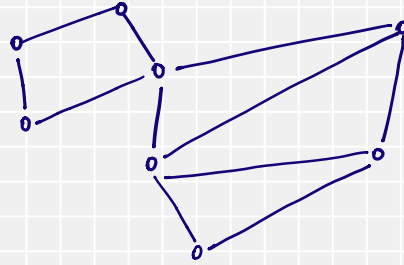


Start with
your favorite
subgraph on
 $k+2$ vertices

↓
Has an "easy"
vertex cover
of size $\leq k+1$

(ANY subset of
size $\leq k+1$ will do!)

* requires the problem to be hereditary



Feedback Vertex Set

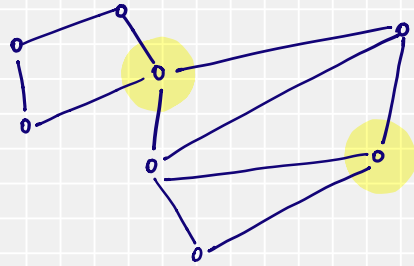
Input: A graph G & a budget k

Question: $\exists S \subseteq V$ such that $G \setminus S$ is a forest
& $|S| \leq k$?

Parameter: k

0
0
0

KILL
all cycles!



Feedback Vertex Set

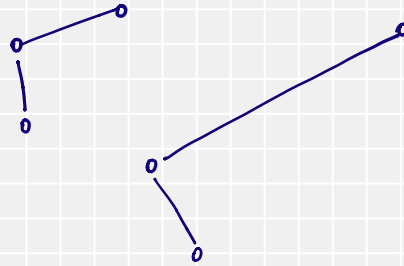
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Feedback Vertex Set

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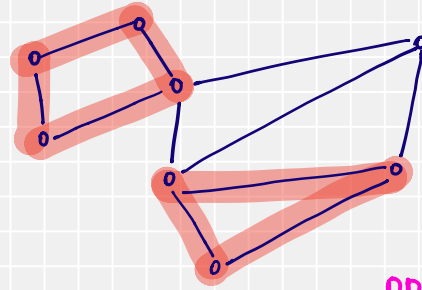
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Parameter: k

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0

KILL
all cycles!

Feedback Vertex Set



opt FVS = 2

Input: A graph G & a budget k

Question: $\exists S \subseteq V$ such that $G \setminus S$ is a forest
& $|S| \leq k$?

Parameter: k

0
0
0

KILL
all cycles!

The Compression Edition

Feedback Vertex Set

& a FVS of size $k+1$

Input: A graph G & a budget k



Question: $\exists S \subseteq V$ such that $G \setminus S$ is a forest
& $|S| \leq k$?

Parameter: k

0
0
0

KILL
all cycles!

The Compression Edition

Feedback Vertex Set

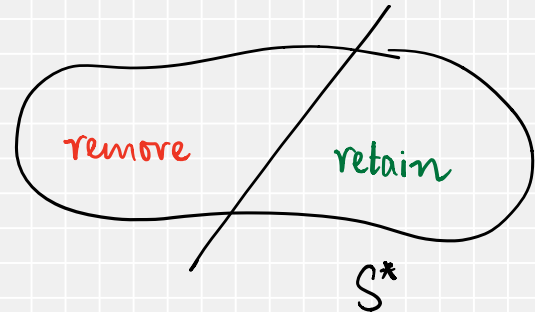
S^*
& a FVS of size $k+1$

Input: A graph G & a budget k



Question: $\exists S \subseteq V$ such that $G \setminus S$ is a forest
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Parameter: k



guess
this!

The Compression Edition

Feedback Vertex Set

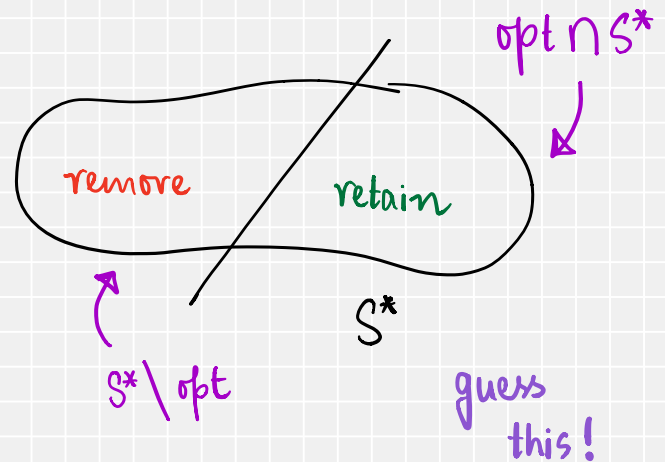
S^*
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The Compression Edition

Feedback Vertex Set

S^*
& a FVS of size $k+1$

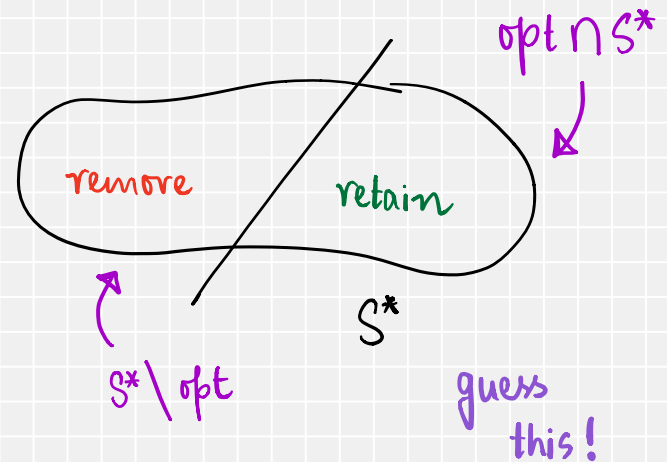
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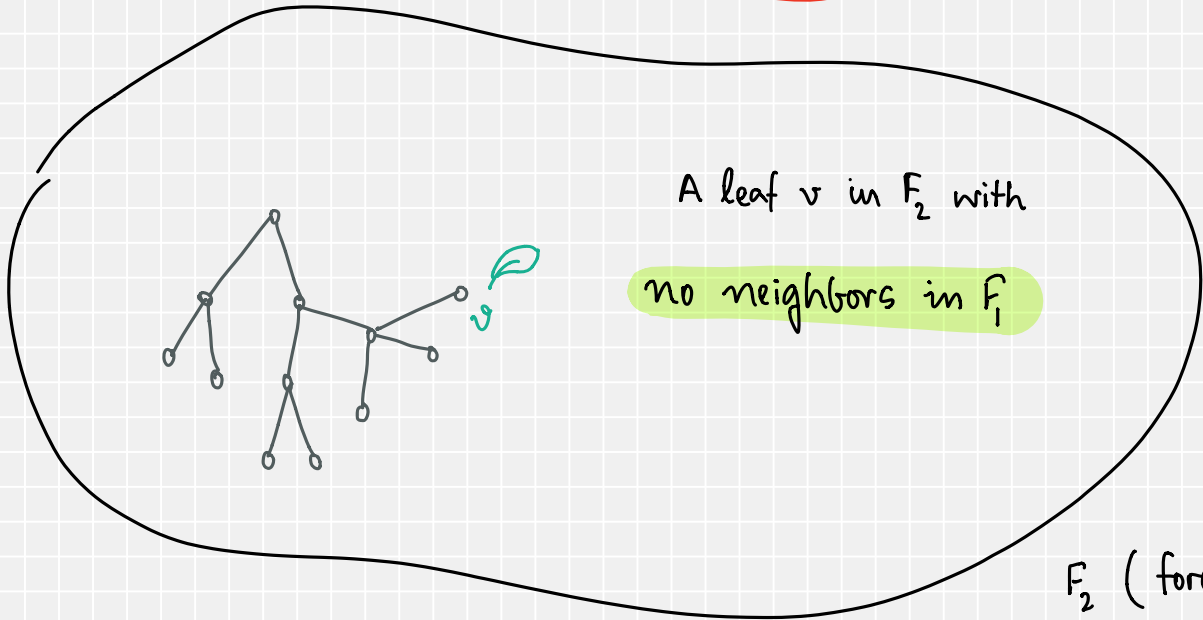
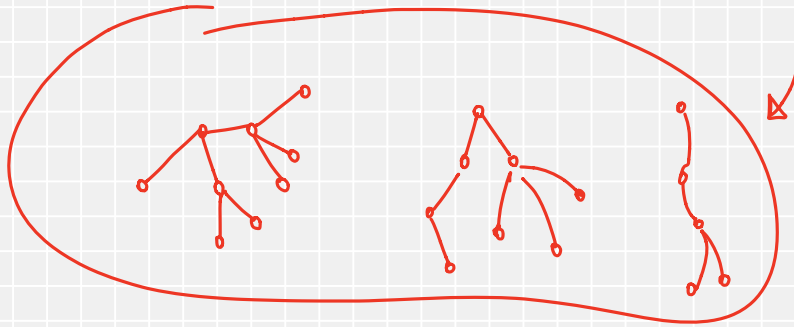
Observe that $S^* \setminus \text{opt}$ should be acyclic



Disjoint fvs

forbidden vertices

(forest) F_1

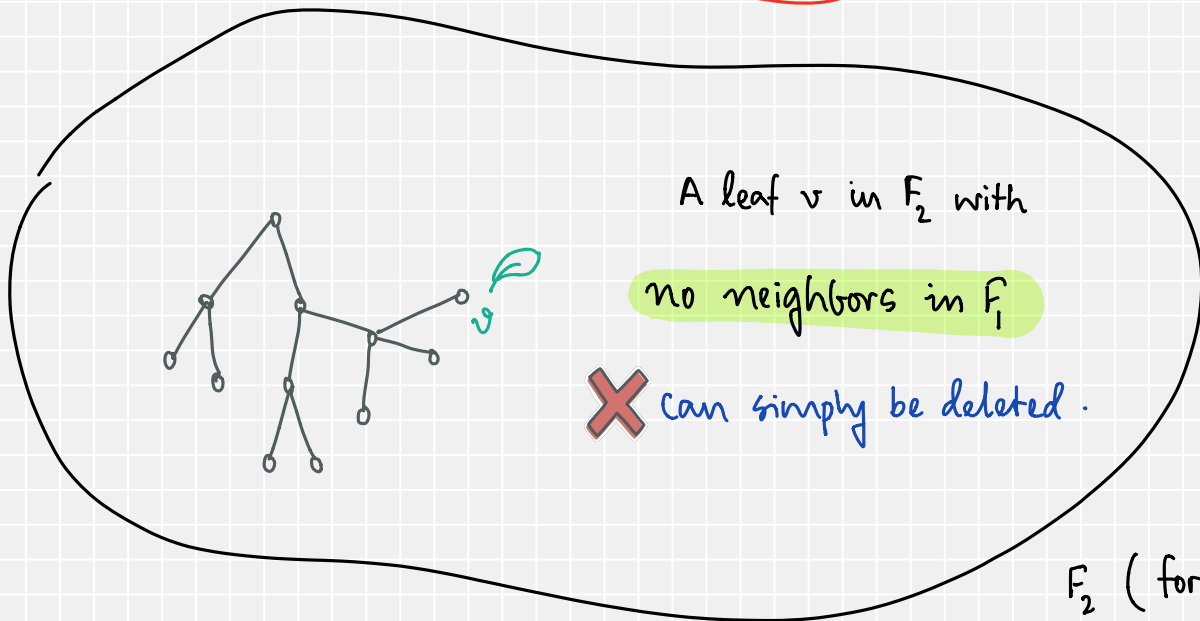
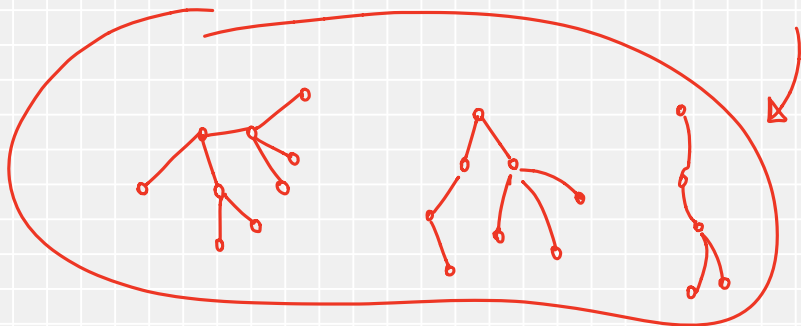


F_2 (forest)

Disjoint fcs

forbidden vertices

(forest) F_1



A leaf v in F_2 with

no neighbors in F_1

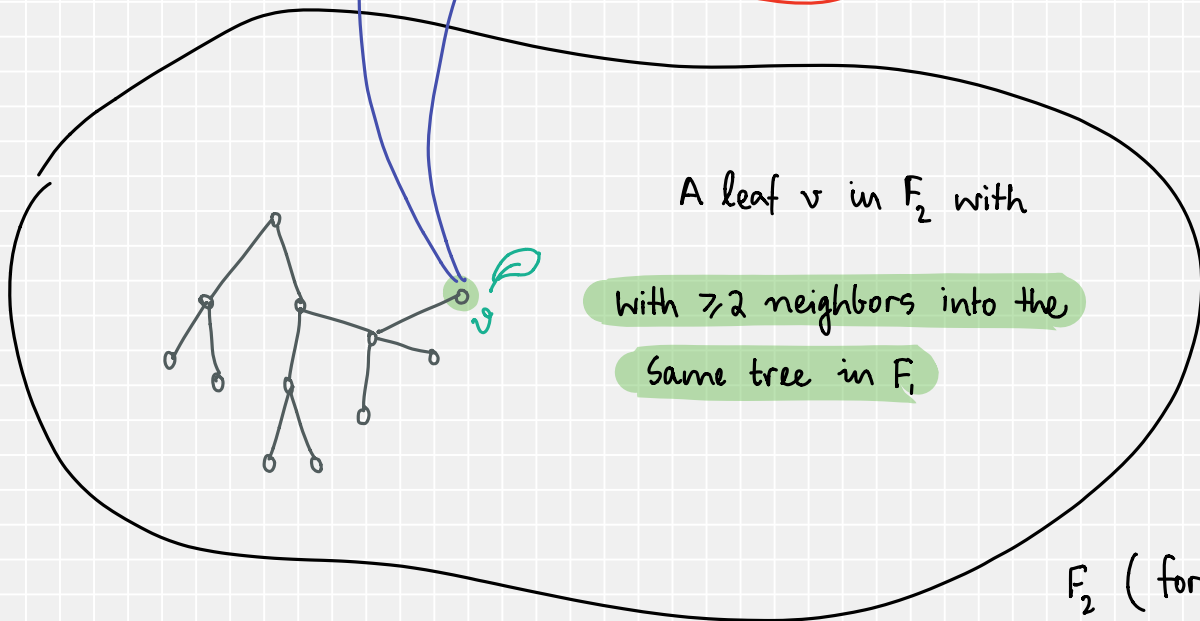
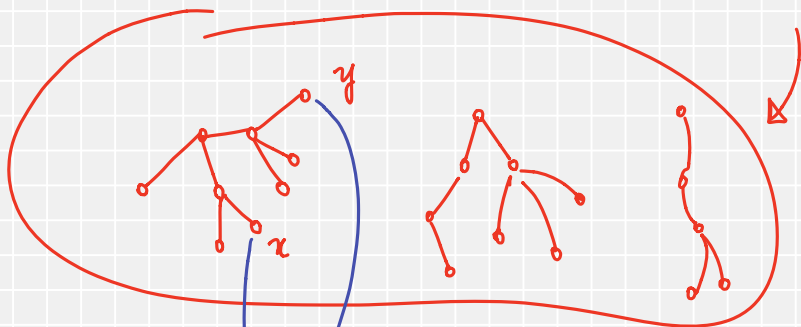
X can simply be deleted.

F_2 (forest)

Disjoint fcs

forbidden vertices

(forest) F_1

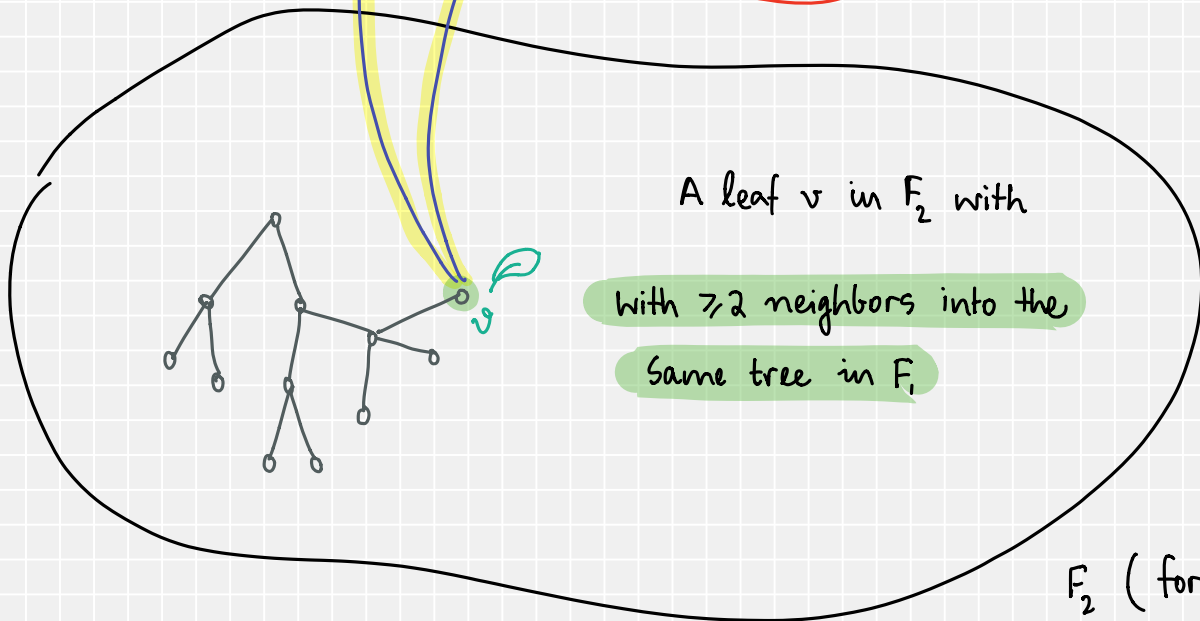
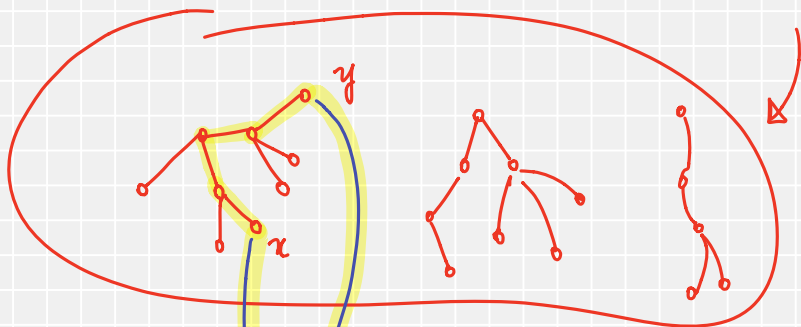


F_2 (forest)

Disjoint fcs

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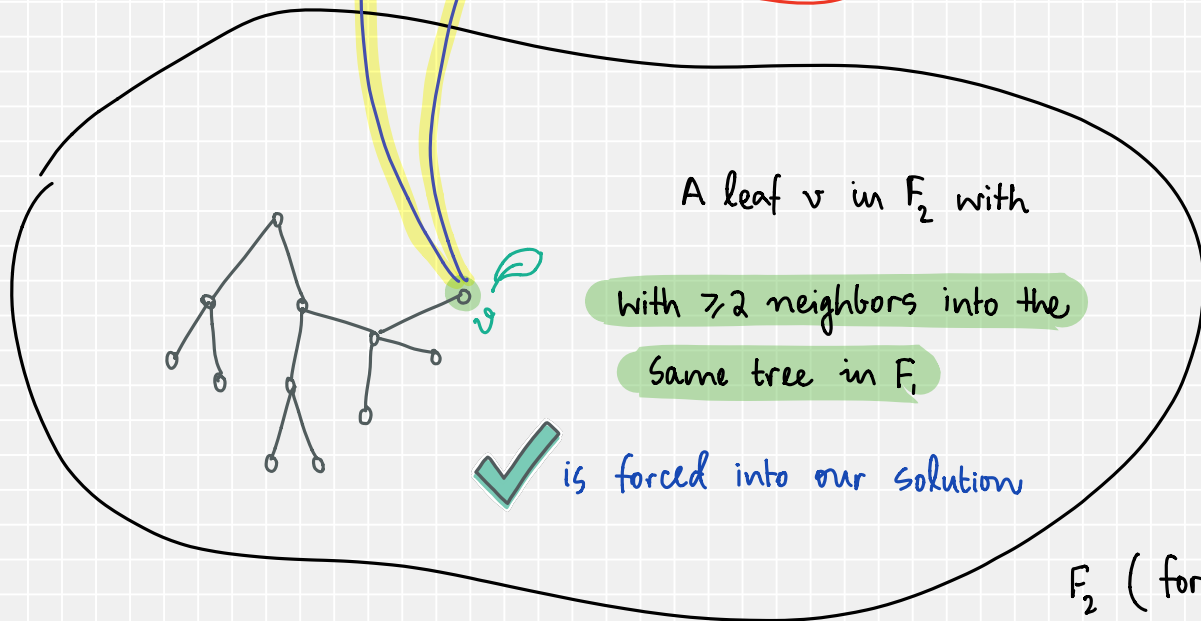
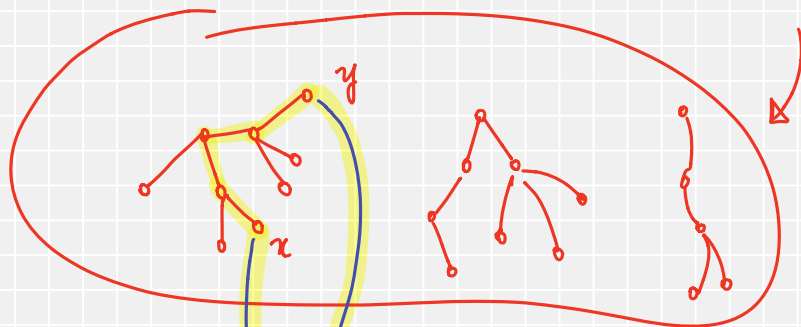
(forest) F_1



Disjoint fcs

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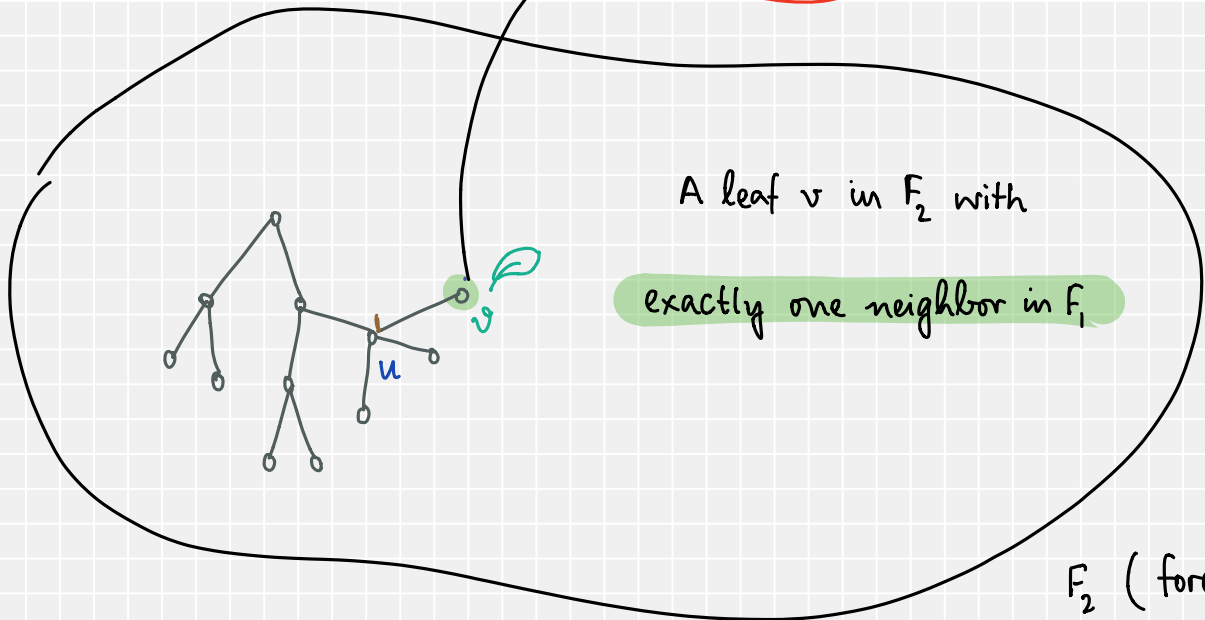
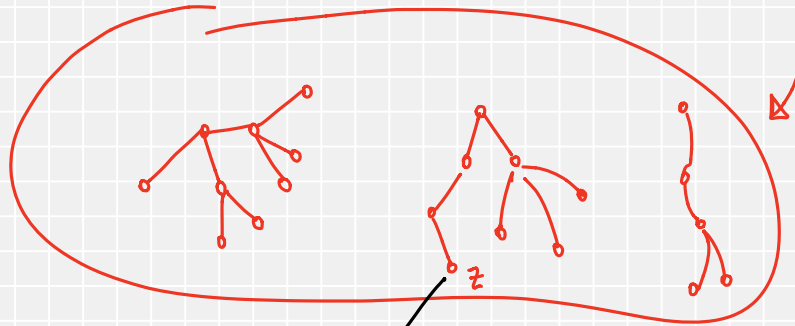
(forest) F_1



Disjoint fvs

forbidden vertices

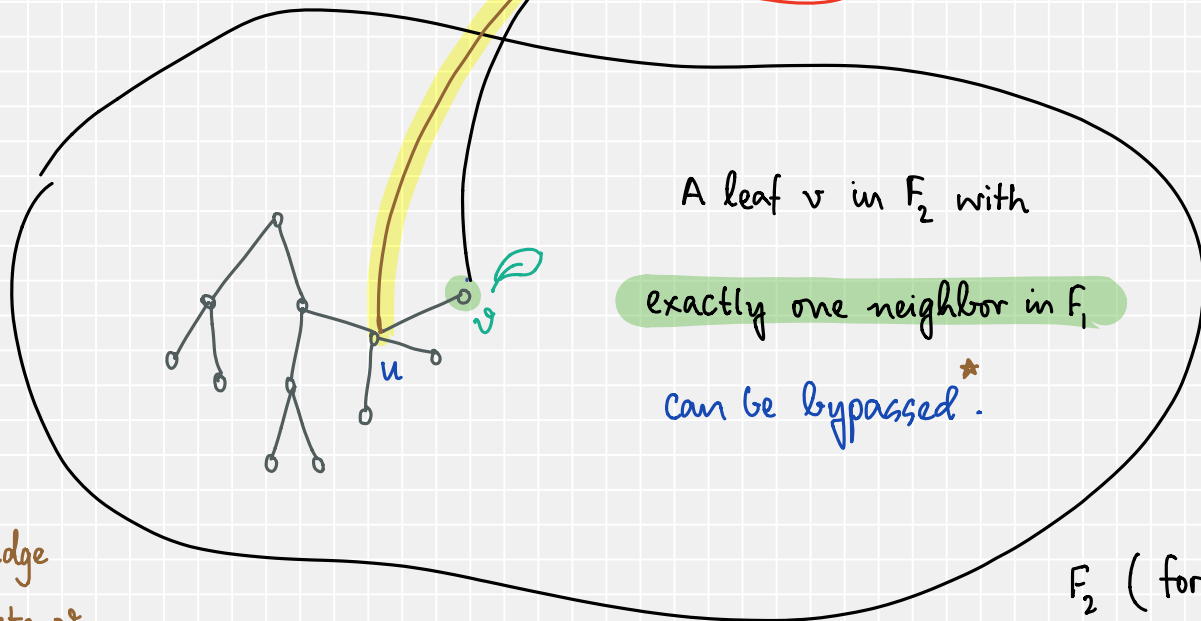
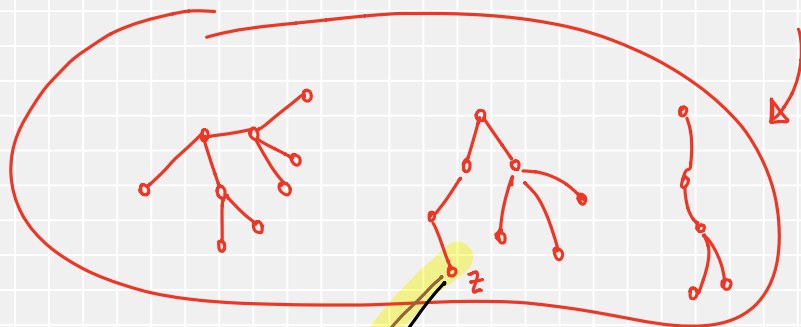
(forest) F_1



Disjoint fRS

forbidden vertices

(forest) F_1



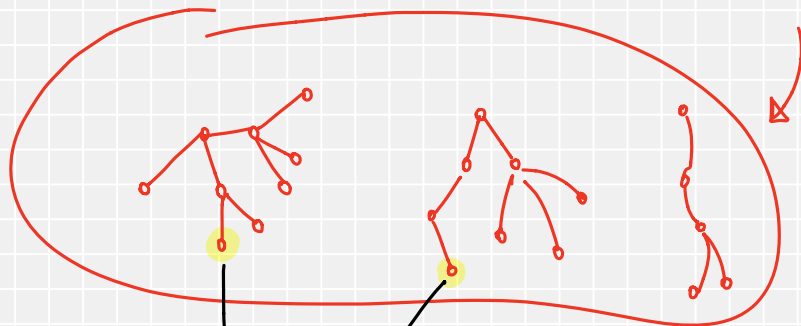
* add the edge
(z, u) & delete v

F_2 (forest)

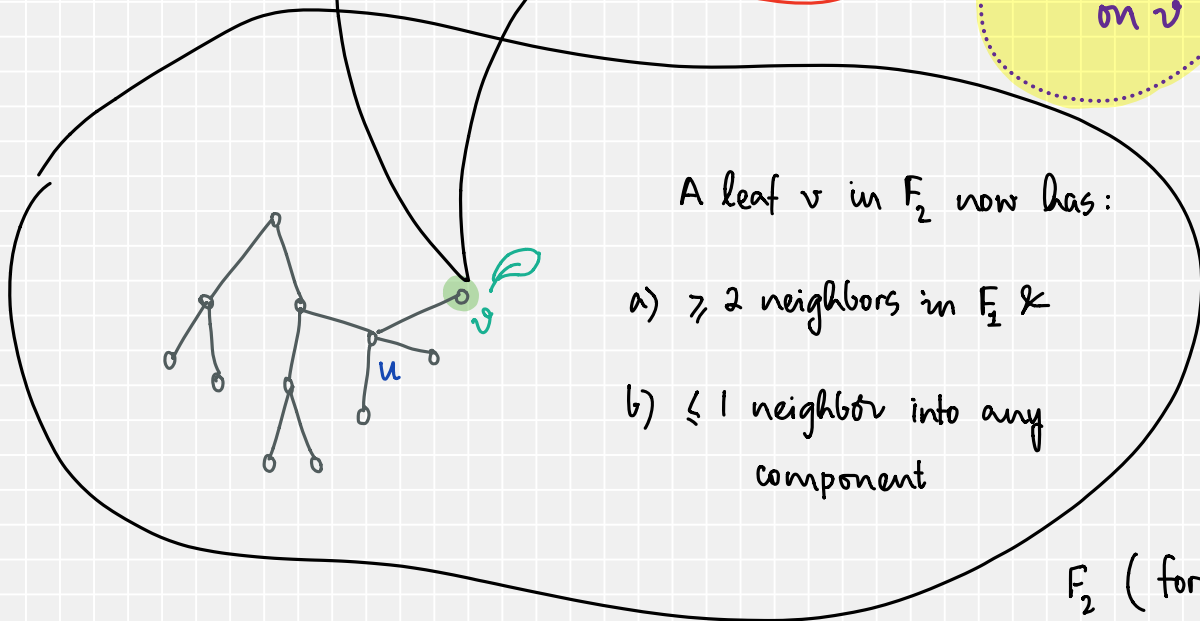
Disjoint fcs

forbidden vertices

(forest) F_1



Branch
on v



A leaf v in F_2 now has:

- a) ≥ 2 neighbors in F_1 &
- b) ≤ 1 neighbor into any component

F_2 (forest)

Disjoint fvs

forbidden vertices

(forest) F_1

$v \in FVS$



budget
reduces

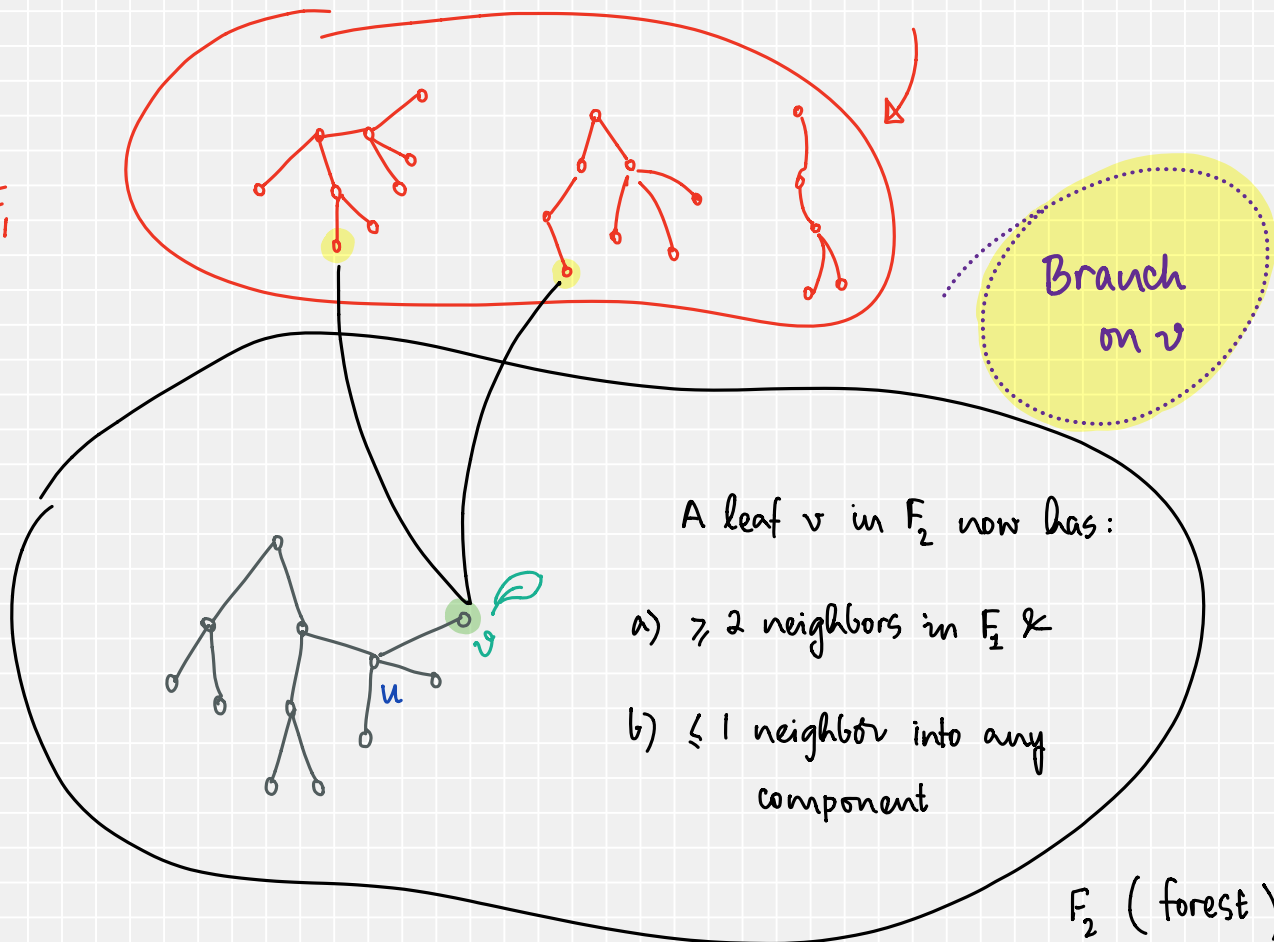
$(G-v, b-1)$

Branch
on v

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F_2 (forest)



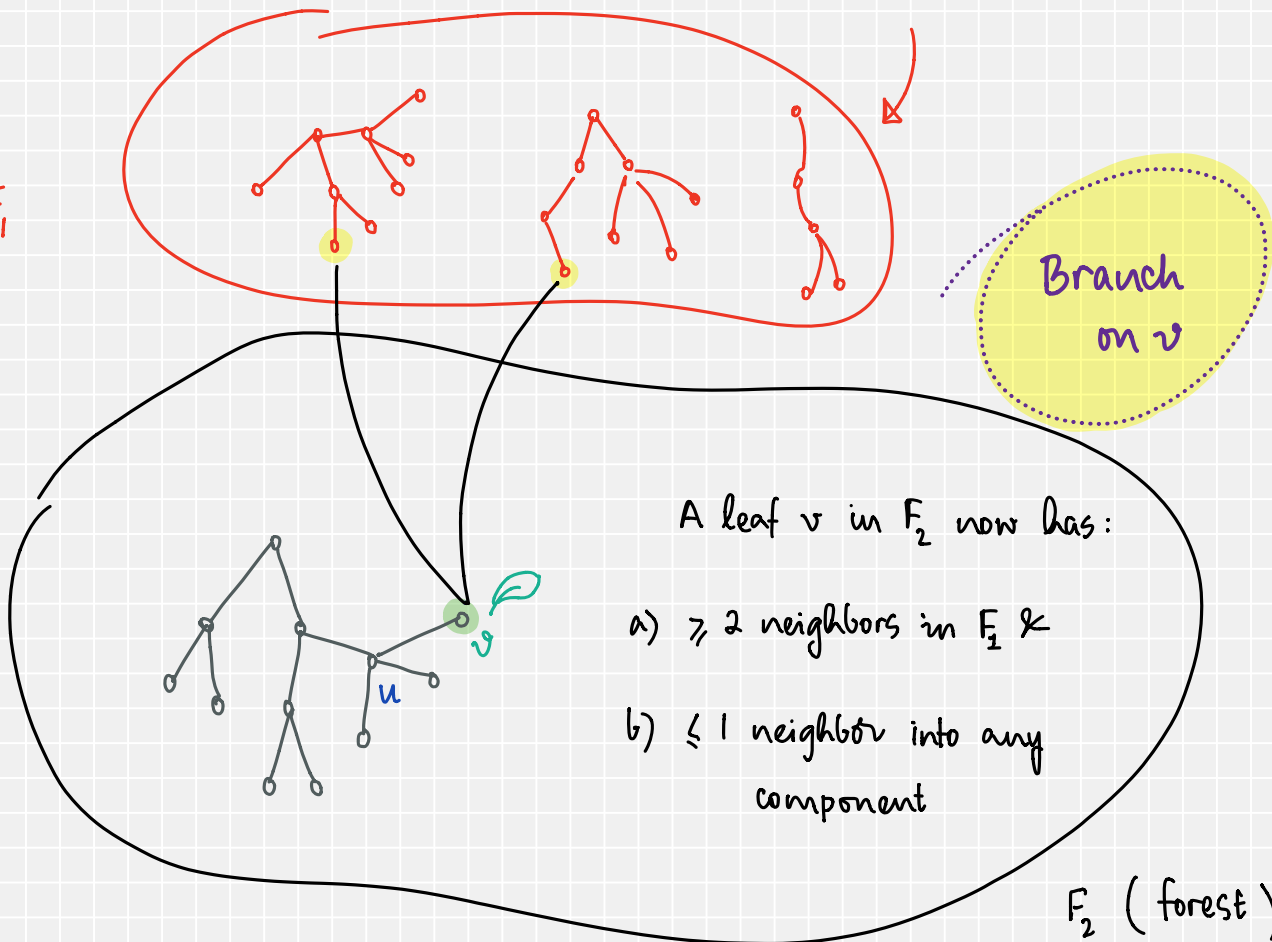
Disjoint fvs

(forest) F_1

forbidden vertices

Branch
on v

$v \notin FVS$



Disjoint fvs

forbidden vertices

(forest) F_1

$v \notin FVS$

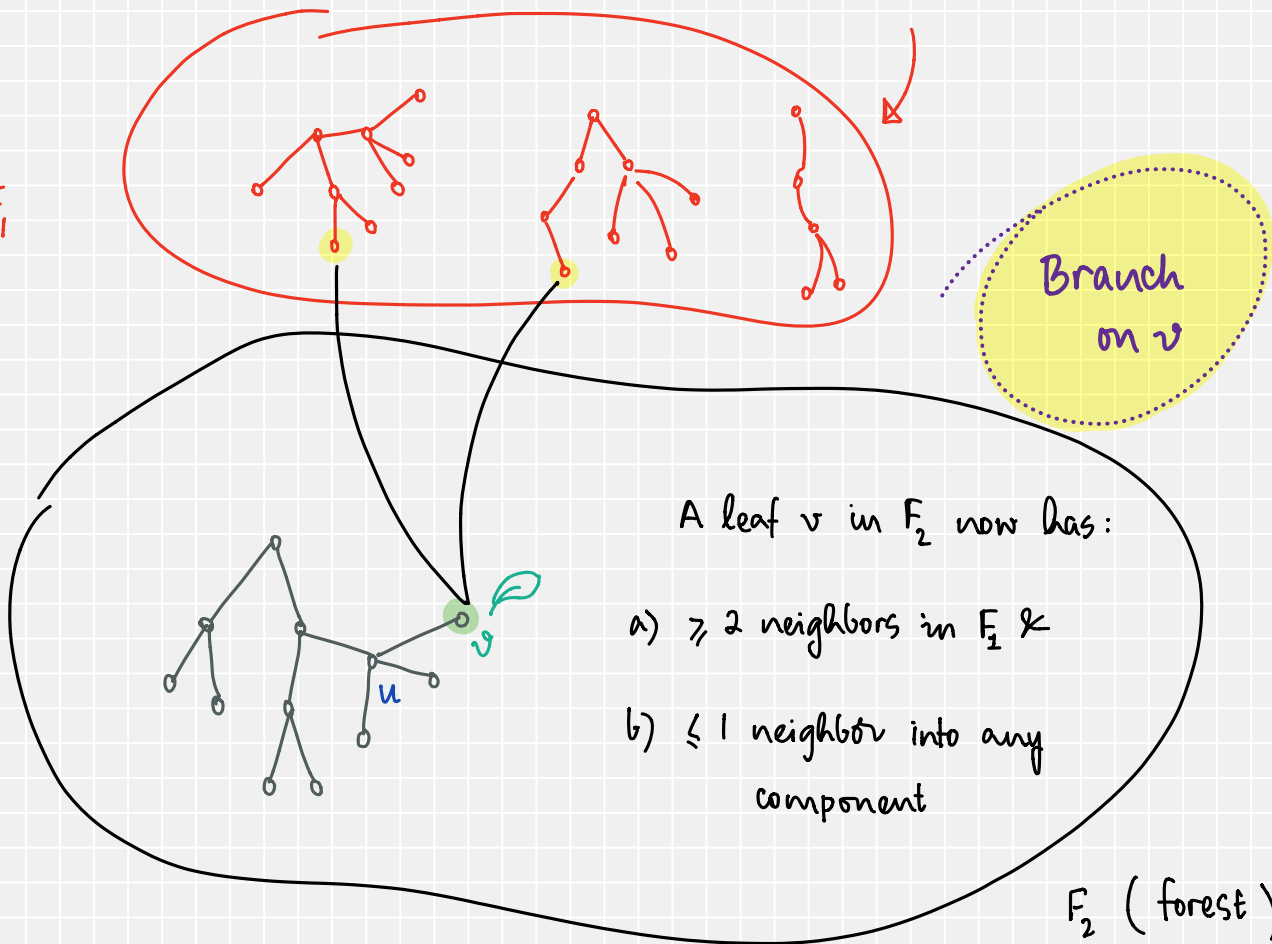
add v
to F_1 .

Branch
on v

A leaf v in F_2 now has:

- a) ≥ 2 neighbors in F_1 &
- b) ≤ 1 neighbor into any component

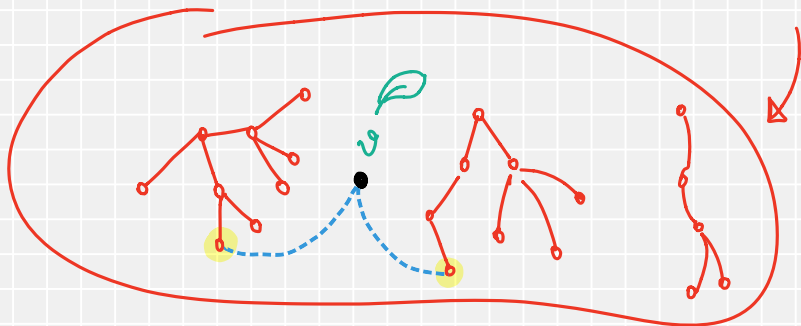
F_2 (forest)



Disjoint fvs

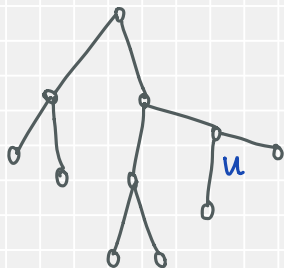
forbidden vertices

(forest) F_1



$v \notin FVS$

add v
to F_1 .



A leaf v in F_2 now has:

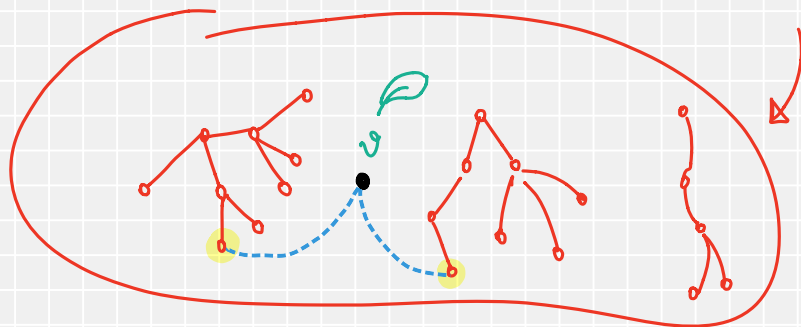
- a) ≥ 2 neighbors in F_1 &
- b) ≤ 1 neighbor into any component

F_2 (forest)

Disjoint fvs

forbidden vertices

(forest) F_1

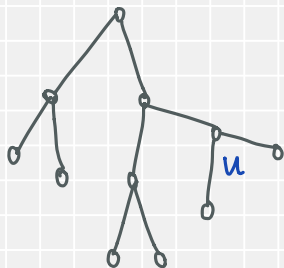


Branch
on v

$v \notin FVS$

add v
to F_1 .

trees in F_1
reduces by 1.



A leaf v in F_2 now has:

- a) ≥ 2 neighbors in F_1 &
- b) ≤ 1 neighbor into any component

F_2 (forest)

$$\mu(F_1, F_2) = \text{budget} + \# \text{ trees in } F_1 \leq 2k$$

reduces
when $v \in FVS$

$$\mu(F_1, F_2) = \text{budget} + \# \text{ trees in } F_1$$

reduces
when $v \in FVS$

$$\mu(F_1, F_2) = \text{budget} + \# \text{ trees in } F_1$$

reduces when
 $v \notin FVS$.

reduces
when $v \in FVS$

$$\mu(F_1, F_2) = \text{budget} + \# \text{ trees in } F_1 \leq 2k$$

reduces when
 $v \notin FVS$.

reduces
when $v \in FVS$

$$\mu(F_1, F_2) = \text{budget} + \# \text{ trees in } F_1 \leq 2k$$

overall running time

$$O(4^k)$$

reduces when
 $v \notin FVS$.

BACK TO COMPRESSION

$$\sum_{\text{remove} \subseteq S^*} \text{DISJOINT-FVS} (F_1 = \text{remove}, F_2 = V \setminus S^*)$$

BACK TO COMPRESSION

$$\sum_{\text{remove} \subseteq S^*} \text{DISJOINT-FVS} (F_1 = \text{remove}, F_2 = V \setminus S^*)$$

$$\sum_{l=1}^k \binom{k+1}{l} 2^{\{l-1 + l\}}$$

↑
max # of trees in F_1

↓
Budget

BACK TO COMPRESSION

$$\sum_{\text{remove} \subseteq S^*} \text{DISJOINT-FVS} (F_1 = \text{remove}, F_2 = V \setminus S^*)$$

$$\sum_{l=1}^k \binom{k+1}{l} 2^{\{ \overset{\substack{\text{max \# of trees in } F_1 \\ \uparrow}}{l-1} + \underset{\substack{\text{Budget} \\ \downarrow}}{l}} \}} \leq \sum_{l=1}^k \binom{k+1}{l} 4^l = O(5^k).$$