

# HITTING AND PIERCING RECTANGLES INDUCED BY A POINT SET

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# INDUCED GEOMETRIC OBJECTS

$P$  - set of  $n$  points in  $\mathbb{R}^2$  in general position.

$\mathcal{R}$  - Set of **all** distinct geometric objects of a particular class induced (spanned) by  $P$ .

For example, let  $\mathcal{R}$  be the set of **all** the  $\binom{n}{2}$  axis-parallel rectangles induced by a distinct pair of points in  $P$ .

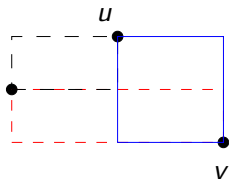


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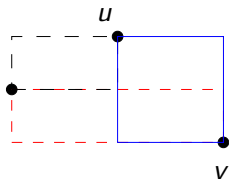


**FIGURE:** Set of all axis-parallel Rectangles induced by  $P$

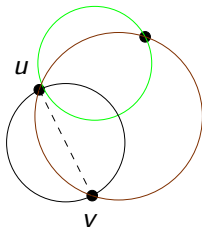


**FIGURE:** Set of all diametrical Disks induced by  $P$

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Broadly, we look at 2 kinds of problems in this paper

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(Selection Lemma)
- 2 What is the minimum set of points needed to hit all the objects in  $\mathcal{R}$ ?  
(Hitting Set)

# FIRST SELECTION LEMMA (FSL)

- 1 For induced triangles in  $\mathbb{R}^2$ , Boros and Füredi (1984), showed that the centerpoint is present in  $\frac{n^3}{27}$  (constant fraction) triangles induced by  $P$ . This constant is tight. A classical result in discrete geometry.

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- ② For induced simplices in  $\mathbb{R}^d$ , Bárány (1982) showed that there exists a point  $p \in \mathbb{R}^d$  contained in at least  $c_d \cdot \binom{n}{d+1}$  simplices induced by  $P$ . Result used in the construction of weak  $\epsilon$ -nets for convex objects (Matousek 2002).

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- ③ FSL type results have not been explored for other classes of induced objects like axis-parallel rectangles, disks etc.
- ④ Strong first selection lemma ( $p \in P$ ).

# SECOND SELECTION LEMMA (SSL)

- 1 Generalization of the first selection lemma, which considers an  $m$ -sized arbitrary subset  $\mathcal{S} \subseteq \mathcal{R}$  and shows that there exists a point which is contained in  $f(m, n)$  objects of  $\mathcal{S}$ .

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- ⑤ Shakhar et al.(2004) gave an alternate proof of the same bounds and also gave an upper bound of  $O(\frac{m^2}{n^2 \log(\frac{n^2}{m})})$ .

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- 2 We explore these questions for special cases of induced axis-parallel rectangles like skylines, slabs etc.
- 3 Combinatorial **Bounds** studied for induced disks, axis-parallel rectangles and triangles.

# OUR RESULTS

## 1 Selection Lemmas

- ① For the first selection lemma for axis-parallel rectangles, we show a tight bound of  $\frac{n^2}{8}$ .
- ② For the strong first selection lemma for axis-parallel rectangles, we show a tight bound of  $\frac{n^2}{16}$ .
- ③ (Second selection lemma) We show that  $f(m, n) = \Omega(\frac{m^3}{n^4})$  for axis-parallel rectangles. Improvement over the previous bound in Shakh et al.(2004), when  $m = \Omega(\frac{n^2}{\log^2 n})$ .

## 2 Hitting set for induced objects

- ① The hitting set problem for all induced lines is NP-complete.
- ② Induced axis-parallel skyline rectangles.
  - ①  $O(n \log n)$  time algorithm to compute the minimum hitting set.
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Some notation -

- ①  $P$  - Pointset of size  $n$  in  $\mathbb{R}^2$  in general position.
- ②  $R(u, v)$  - axis-parallel rectangle induced by  $u$  and  $v$  where  $u, v \in P$ .
- ③  $\mathcal{R}$  - set of all  $R(u, v)$  for all  $u, v \in P$ .
- ④  $R_p \subset \mathcal{R}$  - set of axis-parallel rectangles that contain  $p \in \mathbb{R}^2$ .

## HIGH LEVEL IDEA OF THE PROOF

## THEOREM

Let  $f(n) = \min_{P, |P|=n} (\max_{p \in \mathbb{R}^2} |R_p|)$ .

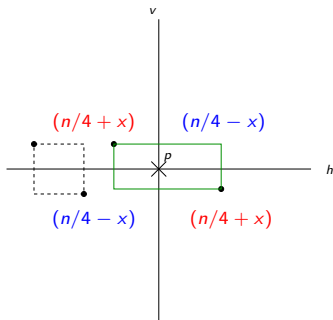
*There exists a point  $p$  in  $\mathbb{R}^2$  (not necessarily belonging to  $P$ ), which is present in at least  $\frac{n^2}{8}$  axis-parallel rectangles induced by  $P$  i.e  $f(n) \geq \frac{n^2}{8}$ . This bound is tight.*

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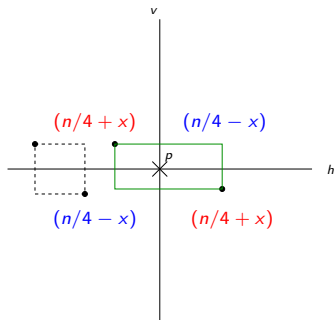


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Horizontal line  $h$  and vertical line  $v$ , each of which bisects the pointset.

$$|R_p| = \left(\frac{n}{4} - x\right)^2 + \left(\frac{n}{4} + x\right)^2$$

$$\implies |R_p| = \frac{n^2}{8} + 2x^2$$

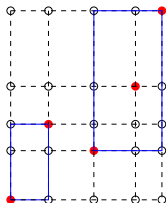
$$\text{Thus, } |R_p| \geq \frac{n^2}{8}.$$

# THE PROBLEM

**The problem** - Let  $\mathcal{S} \subseteq \mathcal{R}$ ,  $|\mathcal{S}| = m$ . We bound the maximum number of rectangles in  $\mathcal{S}$  that can be pierced by a single point  $p \in \mathbb{R}^2$ .

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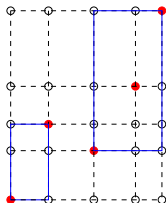
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Construct a grid out of  $P$ . Let the grid points be  $G$  ( $P \subset G$ ), where  $|G| = n^2$ .  $G$  is the candidate set of points for the second selection lemma.

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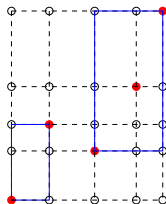


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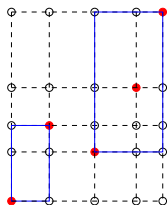
*If  $m = \Omega(n^{\frac{4}{3}})$ , there exists a point  $p \in G$  which is present in at least  $\frac{m^3}{24n^4}$  rectangles of  $\mathcal{S}$ .*

# SKETCH OF THE PROOF



- We find a lower bound for the sum of grid points contained in each rectangle in  $\mathcal{S}$ .
- Same as counting the number of rectangles of  $\mathcal{S}$  pierced by a grid point, summed over all grid points.

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- By pigeonhole principle, we find a lower bound on the rectangles of  $\mathcal{S}$  pierced by some grid point.

# NUMBER OF GRID POINTS IN $X'_i$ - LOWER BOUND

Some notations used in the proof -

- ① The rectangle  $R(x_i, u) \in \mathcal{S}$  where  $x_i, u \in P$  is added to the partition  $X_i$ , if  $u$  is higher than  $x_i$  (similarly  $P_i$ ). Further partitioned into  $X'_i$  and  $X''_i$  (right and left).
- ② Let  $|X'_i| = |P'_i| = m'_i$ .
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Let  $c = \sum_{r \in X'_i} J_r$ . Then  $c \geq \frac{(m'_i)^3}{6}$ .

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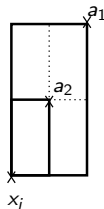
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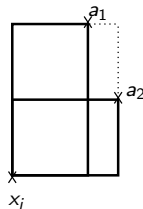
Proof is by induction on  $m'_i$ .

## BASE CASE

- **Base Case,  $m'_i = 2$**



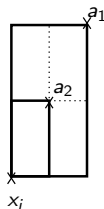
(i)



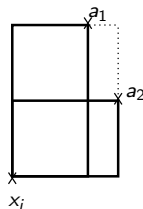
(ii)

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(i)

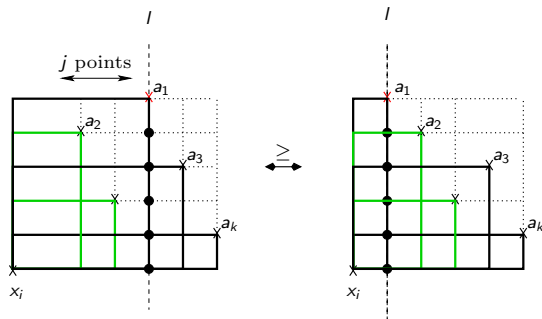


(ii)

- Assume that the statement is true for  $m'_i = k - 1$  and let  $m'_i = k$ .
- We prove that the lower bound is achieved when  $P'_i$  is monotonically decreasing and any other pointset is worse than this.

## INDUCTIVE HYPOTHESIS - CASE 1

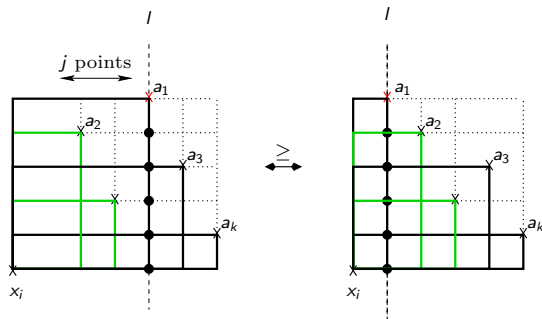
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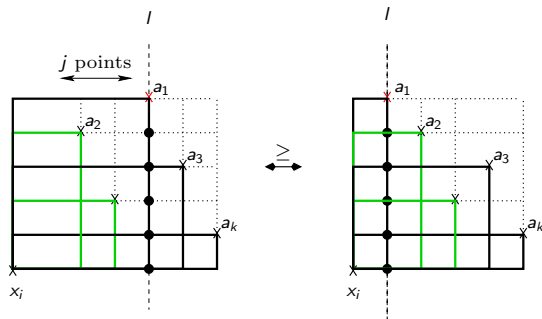
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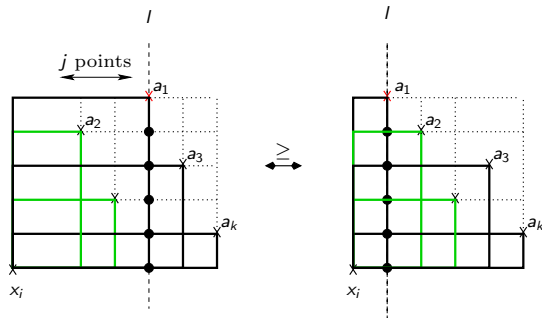
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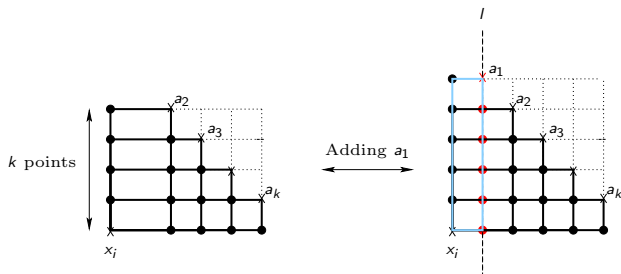
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- Thus, we see that  $c$  does not increase.

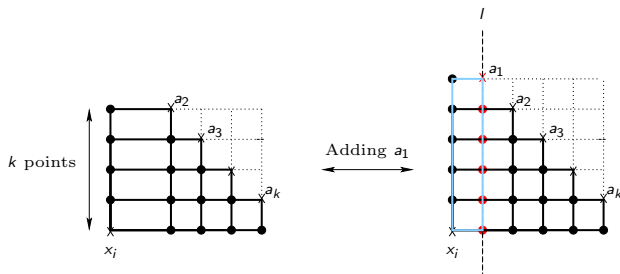
Case 2 :  $a_1$  is the leftmost point.



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# INDUCTIVE HYPOTHESIS - CASE 2

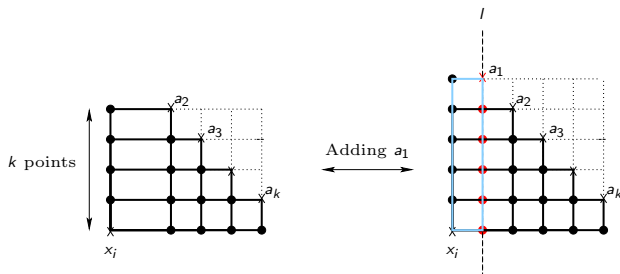
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- By summing all these quantities, we see that the induction hypothesis is true.

## PROOF OF THEOREM 2

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- We use an averaging argument ( $n^2$  grid points) and prove the theorem.

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- Can the hitting set for the set of all objects (disks, axis-parallel rectangles etc.) induced by  $P$ , be computed in polynomial time ?